

# UNSTABLE LOCI IN FLAG VARIETIES AND VARIATION OF QUOTIENTS

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**ABSTRACT.** We consider the action of a semisimple subgroup  $\hat{G}$  of a semisimple complex group  $G$  on the flag variety  $X = G/B$ , and the linearizations of this action by line bundles  $\mathcal{L}$  on  $X$ . The main result is an explicit description of the associated unstable locus in dependence of  $\mathcal{L}$ , as well as a combinatorial formula for its dimension. We observe that this codimension tends to be high in the interior of the  $\hat{G}$ -ample cone, while close to the boundary it is low, even shown to equal to 1 on the so called cohomological faces. We also give an algorithm for recursively determining the GIT-classes in the  $\hat{G}$ -ample cone of  $X$ . For our strongest statements, we impose hypothesis on  $\hat{G} \subset G$ , allowing for instance diagonal embeddings and principal subgroups.

As an application, we give conditions ensuring the existence of GIT-classes  $C$  with an unstable locus of codimension at least two and which moreover yield geometric GIT quotients. Such quotients  $Y_C$  reflect global information on  $\hat{G}$ -invariants. They are always Mori dream spaces, and the Mori chambers of the pseudoeffective cone  $\overline{\text{Eff}}(Y_C)$  correspond to the GIT-chambers of the  $\hat{G}$ -ample cone of  $X$ . Moreover, all rational contractions  $f : Y_C \dashrightarrow Y'$  to normal projective varieties  $Y'$  are induced by GIT from linearizations of the action of  $\hat{G}$  on  $X$ . In particular, this is shown to hold for a diagonal embedding  $\hat{G} \hookrightarrow (\hat{G})^k$ , with sufficiently large  $k$ .

## CONTENTS

1. Introduction	2
2. Setting and statement of the main results	4
3. The Hilbert-Mumford criterion	7
4. Regular one-parameter subgroups	8
4.1. Compatible Weyl chambers and cubicles	11
5. A formula for the unstable locus	11
6. The case of single cubicle	13
6.1. The $\hat{G}$ -ample and movable cones	14
6.2. GIT-classes for $\hat{T}$ and $\hat{G}$	15
6.3. Some general bounds on the codimension of the unstable locus	18
6.4. Cohomological components and Lie algebra cohomology	19
7. Mori chambers	22
References	26

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## 1. INTRODUCTION

One of the fundamental problems in representation theory, occurring in various situations, is the understanding of the space of invariants  $V^{\hat{G}}$ , where  $\hat{G} \rightarrow G$  is a morphism of groups and  $V$  is a representation space of  $G$ . We apply the framework of Geometric Invariant Theory with Variations (VGIT in the sense of [DH98],[T96]) to embeddings of semisimple complex algebraic groups  $\iota : \hat{G} \subset G$  and study comparatively the  $\hat{G}$ -action on the complete flag variety  $X = G/B$  with  $B$  a Borel subgroup, on one hand, and the variations in the dimension of the space of invariants  $V^{\hat{G}}$  for finite dimensional irreducible  $G$ -modules  $V$ , on the other hand. The setting is classical, developed to a large extent in works related originally to the Horn problem formulated in terms of decompositions of tensor products of representations, here  $GL_n = \hat{G} \xrightarrow{\text{diag}} G = GL_n^{\times k}$ , see [BKR12] for a modern account. The general branching problem of decomposing  $G$ -modules over  $\hat{G}$  can be formulated as the problem of invariants for the diagonal embedding  $\text{id} \times \iota : \hat{G} \subset \hat{G} \times G$ , via the isomorphism  $\text{Hom}_{\hat{G}}(\hat{V}, V) \cong (\hat{V}^* \otimes V)^{\hat{G}}$ . Recalling that the irreducible  $G$ -modules are parametrized up to isomorphism by the elements of a Weyl chamber  $\Lambda^+$  in the character lattice  $\Lambda$  of a maximal torus  $T \subset B$ , one is led to consider the so-called (generalized) Littlewood-Richardson monoid and cone, also known as the eigencone, or in the GIT-context adopted here, the  $\hat{G}$ -ample cone on  $X$ :

$$LR(\hat{G} \subset G) = \{\lambda \in \Lambda^+ : V_\lambda \neq 0\}, \quad C^{\hat{G}}(X) = \text{Span}_{\mathbb{R}_+} LR \subset \Lambda_{\mathbb{R}},$$

They are shown by Brion and Knop to be indeed a finitely generated monoid and a rational polyhedral cone. Our results concern the finer structures in  $C^{\hat{G}}(X)$  induced by VGIT and the global behaviour of the multiplicities, as well as the development of geometric and combinatorial methods for the study of these structures. As an outcome we exhibit a family of Mori dream spaces obtained as quotients, containing global information on the invariants, and presenting a potential interest for Mori theory and its interactions with the structure theory of semisimple groups.

The GIT approach to the problem of finding invariants for a given  $\iota$  has been developed and applied successfully in a series of works, notably by Heckman, Berenstein, Sjamaar, Belkale, Kumar, Ressayre, Richmond and others (see [H82], [BS00], [BK06], [R10], [RR11] and the references therein), culminating in a description of  $C^{\hat{G}}(X)$  by a minimal set of inequalities. Without citing full statements, we sketch some aspects of the solution, since this will help introduce the setting and put our work into context. The first milestone is the Borel-Weil theorem, providing models for the irreducible  $G$ -modules as the spaces of section of effective line bundles on the flag variety,  $H^0(X, \mathcal{L}_\lambda) = V_\lambda^*$  for  $\lambda \in \Lambda^+$ , with the isomorphism of lattices  $\text{Pic}(X) \cong \Lambda$ , given by  $\mathcal{L}_\lambda = G \times_B \mathbb{C}_{-\lambda}$ ; the dominant Weyl chamber  $\Lambda^+$  spans the pseudoeffective cone, while the ample line bundles are given by strictly dominant weights  $\Lambda^{++}$ . The  $\hat{G}$ -ample cone in  $\text{Pic}(X)_{\mathbb{R}}$  is then given by the line bundles admitting nonconstant invariants in their section rings. This construction fits into the framework of GIT, [FMK94], thus providing tools, in particular the Hilbert-Mumford criterion, for the study of the  $\hat{G}$ -action and invariants. The inequalities defining  $C^{\hat{G}}(X)$  in  $\Lambda_{\mathbb{R}}$ , given in [BS00] and optimized in later works, are derived from the Hilbert-Mumford criterion, and have the form

$$(1) \quad \lambda(w\xi) \leq 0,$$

where  $\lambda$  is the dominant weight representing the line bundle, while  $w$  and  $\xi$  are, respectively, an element of the Weyl group  $W$  of  $G$  and  $\xi$  a one-parameter subgroup in a Cartan subgroup of the form  $\hat{T} = \hat{G} \cap T$ . The pairs  $w, \xi$  appearing in the inequalities are subject to certain conditions, and the description of these conditions

presents the main technical issue. A finite but redundant list of pairs is given by Berenstein and Sjamaar, minimized by Belkale and Kumar in the diagonal case, and by Ressayre for arbitrary embeddings of reductive groups. The list of relevant elements  $\xi$  is relatively easy to obtain, they are determined by the weights of the  $\hat{G}$ -action on the quotient of Lie algebras  $\mathfrak{g}/\hat{\mathfrak{g}}$ ; in the diagonal case, one has simply the fundamental coweights of  $\hat{G}$ . The relevant Weyl group elements present a more delicate problem. The conditions on  $w$ , given in the aforementioned series of works starting with [BS00], are cohomological, stated in terms of pullbacks of Schubert classes from flag varieties of  $G$  to closed  $\hat{G}$ -orbits in them. There is an interest in a cohomology-free description of the  $\hat{G}$ -ample cone, and this has been achieved for diagonal embeddings in [BK06] with a non optimal list, optimized for some classical groups in term of quiver representations by Derksen-Weyman [DW11] for groups of type A, and Ressayre [R12] in types A,B,C.

We obtain a cohomology free description of  $C^{\hat{G}}(X)$  by a finite list of inequalities, redundant in general, as a biproduct of one of our main results, under the hypothesis that  $\hat{G}$  contains regular elements of  $G$ . A formulation is given in Theorem 6.5 under a stronger hypothesis, which can be relaxed using Theorem 5.2. Both hypotheses include diagonal embeddings, which we use as a source of examples. The proof is in fact parallel to that of [BS00], the difference being rather formal than essential, but we present a full argument based directly on the Hilbert-Mumford criterion. Our goal is in fact the structure behind the boundary of  $C^{\hat{G}}(X)$ .

For a view on the global behaviour of invariants, it is convenient to consider the Cox ring of  $X$ , or total coordinate ring, consisting in this case of the sum of all irreducible  $G$ -modules, with its subrings generated by the individual line bundles. (The ring structure is induced by cup product and requires a fixed basis of line bundles, [HK00], but at this stage we are only considering the vector space.)

$$\text{Cox}(X) = \bigoplus_{\mathcal{L} \in \text{Pic}(X)} H^0(X, \mathcal{L}) \cong \bigoplus_{\lambda \in \Lambda^+} V_{\lambda} \quad , \quad R_{\lambda} = \bigoplus_{j=0}^{\infty} H^0(X, \mathcal{L}_{\lambda}^j) = \bigoplus_{j=0}^{\infty} V_{j\lambda}^* .$$

The  $\hat{G}$ -invariants we are after are then all assembled in the invariant ring  $\text{Cox}(X)^{\hat{G}}$ , which is also finitely generated. Cox rings are an important ingredient in the theory of Mori dream spaces, the latter having finitely generated Cox rings as one of their essential defining properties (cf. [HK00] for the full definition). The flag varieties form indeed a class of known examples. It is natural to ask about a variety, a quotient  $Y$ , with  $\text{Cox}(X)^{\hat{G}}$  as a Cox ring, having the classical result for individual line bundles in mind. Such a variety would be a geometric incarnation of the complete information on invariants for the given pair  $\hat{G} \subset G$ . This topic is addressed in [S14], where such quotients are constructed and shown to be Mori dream spaces. The construction rests, however, on a nontrivial assumption for existence of  $\hat{G}$ -movable chambers in  $C^{\hat{G}}(X)$ , defined by line bundles whose rings of nontrivial invariants have vanishing locus, the unstable locus  $X^{us}(\lambda)$ , of codimension at least 2 in  $X$ , containing all points with positive dimensional stabilizers.

In the first part of this article, we address the question of existence of  $\hat{G}$ -movable chambers. We devise a general method addressing this problem built on a closed formula for the unstable locus, formulated in Theorem I in the next section. This formula allows us to study GIT-classes of line bundles and their variations. We give a description of the GIT-classes, showing that all inequalities defining chambers in  $C^{\hat{G}}(X)$  are of same type as (1), and we provide a recursive procedure arriving at the relevant  $w\xi$ , see Theorem 6.15. The codimension of the unstable locus can be estimated and be shown to grow towards the interior of the  $\hat{G}$ -ample cone and stay low near the regular boundary. Indeed, for diagonal embeddings with sufficiently

many factors, we find unstable loci of codimension 1 on the boundary of  $C^{\hat{G}}(X)$  and  $\hat{G}$ -movable chambers in the interior (the existence fails for 2 factors).

The second main result, formulated as Theorem II, concerns the birational geometry of a quotient  $Y$  by a  $\hat{G}$ -movable chamber. We establish a canonical identification of the GIT-chambers in  $C^{\hat{G}}(X)$  with the Mori chambers in the pseudoeffective cone in  $\text{Pic}(Y)$ .

The two theorems are independent of each other. Together they present a method for the detection of the requested quotients and a description of the correspondence between their Cox rings. The content of the article is explained in the next section.

## 2. SETTING AND STATEMENT OF THE MAIN RESULTS

The basic GIT notions relating the geometry of  $X$  to the invariant rings  $R_{\lambda}^{\hat{G}}$  are the notions of instability, semistability, stability and quotients, [FMK94]. The central role in the first part of this article is played by the  $\hat{G}$ -unstable locus, defined by the vanishing of the nonconstant invariants in the section ring of a given line bundle:

$$X^{us}(\lambda) = X_{\hat{G}}^{us}(\lambda) = Z(J_{\lambda}) \subset X, \quad J_{\lambda} = \bigoplus_{j \geq 1}^{\infty} H^0(X, \mathcal{L}_{\lambda}^j)^{\hat{G}} = \bigoplus_{j \geq 1}^{\infty} (V_{\lambda}^*)^{\hat{G}}.$$

The semistable locus is the complementary open set  $X^{ss}(\lambda) = X_{\hat{G}}^{ss}(\lambda) = X \setminus X^{us}(\lambda)$ . The Hilbert-Mumford criterion (stated in the next section) gives a numerical characterization of the unstable or, equivalently, semistable points, and allows to extend the notions to the  $\mathbb{R}$ -Picard group, i.e.,  $\lambda \in \Lambda_{\mathbb{R}}$ . This yields a characterization of the  $\hat{G}$ -ample bundles by having an unstable locus of positive codimension or, equivalently, a nonempty semistable locus. In some sense this simple choice represents the main difference between our view on  $C^{\hat{G}}(X)$  and the view we see in the articles mentioned above, since the cohomological conditions for  $w$  are obtained from the condition for its Schubert cell to contain semistable points. We focus on instability. Let us notice the following tautological characterization for the  $\hat{G}$ -ample cone:

$$C^{\hat{G}}(X) = \{\lambda \in \Lambda_{\mathbb{R}}^+ : \text{codim}_X X^{us}(\lambda) > 0\} \subset \Lambda_{\mathbb{R}},$$

Our first main result is a closed formula for the  $\hat{G}$ -unstable locus and a combinatorial formula for its dimension, proven in Theorem 5.2, under a hypothesis that  $\hat{G}$  contains regular elements of  $G$ . The formula stated here is proven in Theorem 6.1 and is somewhat less technical, but requires a stronger hypothesis. Recall that the  $T$ -fixed points in  $X$  are parametrized by the elements of the Weyl group  $W$ , as  $X^T = \{x_w = wB, w \in W\}$ ; their  $B$ -orbits give the Schubert cell decomposition  $X = \sqcup_w Bx_w$ .

**Theorem I:** *Suppose all regular semisimple elements of  $\hat{G}$  are regular in  $G$ . Then the  $\hat{G}$ -unstable loci of ample line bundles on  $X$  are  $\hat{G}$ -saturation of unions of Schubert cells, given by the formula*

$$(2) \quad X^{us}(\lambda) = \hat{G} \left( \bigsqcup_{w \in W \setminus W^{0-}(\lambda)} Bx_w \right), \quad \text{for } \lambda \in \Lambda^{++}.$$

where  $W^{0-}(\lambda) = \{w : \lambda(w^{-1}h_j) \leq 0, \forall j\}$ , with  $h_1, \dots, h_l$  denoting the fundamental coweights of  $\hat{G}$  generating the monoid of dominant one-parameter subgroups. The codimensions of the saturated cells  $\hat{G}B$  and the unstable locus are given the formula

$$\begin{aligned} \text{codim}_X \hat{G}Bx_w &= \text{codim}_{B^{-}x_w} \hat{B}^{-}x_w. \\ \text{codim}_X X^{us}(\lambda) &= \min\{\text{codim}_{B^{-}x_w} \hat{B}^{-}x_w : w \in W^{0-}(\lambda)\}. \end{aligned}$$

where  $B^-$ ,  $\hat{B}^-$  are the opposite Borel subgroups.

The full statements include combinatorial formulae for the (co)dimensions. Note that, when  $\hat{G}$  is moved inside the bracket, the union of saturated cells  $X^{us}(\lambda) = \cup \hat{G}B_w$  ceases to be disjoint, but Zariski closure is not required. Thus the decomposition is very similar to the Kirwan-Ness stratification, [Kir84],[N84], but not identical. As an immediate corollary, we obtain a cohomology free description of  $C^{\hat{G}}(X)$  formulated below. Although we manage to remove some redundant inequalities, our description is not optimal. It is, however, exact, and allows us to study the interior of the  $\hat{G}$ -ample cone.

The  $\hat{G}$ -ample cone is subdivided into GIT-equivalence classes, defined by equality of the unstable loci, i.e.,  $\lambda \sim \lambda'$  if and only if  $X^{us}(\lambda) = X^{us}(\lambda')$ . For  $\hat{G}$ -ample line bundles, the projective spectrum of the invariant ring is isomorphic to the GIT-quotient of  $X$  defined by Hilbert's equivalence relation on the semistable locus:

$$Y_\lambda = X^{ss}(\lambda) // \hat{G} \cong \text{Proj}(R_\lambda^{\hat{G}}) \quad , \quad x_1 \sim x_2 \iff \overline{\hat{G}x_1} \cap \overline{\hat{G}x_2} \neq \emptyset \text{ (in } X^{ss}(\lambda) \text{)} .$$

The quotients defined by GIT-equivalent bundles are clearly isomorphic, and we denote  $X_{\hat{G}}^{us}(C) = X_{\hat{G}}^{us}(\lambda)$  and  $Y_C = Y_\lambda$  for a GIT-class  $C \ni \lambda$ . The GIT-classes form a system of cones in  $C^{\hat{G}}(X)$ , and there are rational maps between some of these quotients, depending on relations between the corresponding GIT-classes (cf. [DH98], [T96]).

Some important properties of the quotient are reflected in properties of the unstable locus. Note that the quotient is geometric when the semistable orbits are equidimensional. In particular, one considers the set of infinitesimally free orbits closed in the semistable locus, called the stable locus:

$$X_G^s(\lambda) = \{x \in X_G^{ss}(\lambda) : \hat{G}x \subset X_G^{ss}(\lambda) \text{ closed, } \dim \hat{G}x = 0\} ,$$

where  $\hat{G}x$  denotes the stabilizer of  $x$ . The GIT-class of  $\lambda$  is called a chamber if all semistable points are stable, i.e.,  $X_G^{us}(\lambda)$  contains all points with positive dimensional stabilizer. It is shown in [S14] that, for embeddings of semisimple groups acting on complete flag varieties, the chambers are exactly the full dimensional GIT-classes in  $C^{\hat{G}}(X)$ .

The Picard group of the quotient  $Y_\lambda$  is naturally related to the Picard group of  $X$ , cf. [KKV89]. The relation becomes simpler when the unstable locus does not contain divisors. This motivates the definition of  $\hat{G}$ -movable GIT-classes as those whose unstable locus has codimension at least 2. The union of these classes forms a cone, called the  $\hat{G}$ -movable cone on  $X$ , denoted by

$$\text{Mov}^{\hat{G}}(X) = \{\lambda \in \Lambda_{\mathbb{R}}^+ : \text{codim}_X X_G^{us}(\lambda) \geq 2\} \subset C^{\hat{G}}(X) .$$

A  $\hat{G}$ -movable chamber is then a full dimensional GIT-class  $C$  with  $\text{codim}_X X^{us}(C) \geq 2$  and  $X^{ss}(C) = X^s(C)$ . In such a case, we obtain a geometric quotient  $Y_\lambda$  whose Picard group embeds, via pullback followed by extension, as a sublattice of full rank in the Picard group of  $X$ , yielding an isomorphism over the reals. Such a quotient is shown in [S14] to be a Mori dream space, whose effective cone is identified with  $C^{\hat{G}}(X)$ . **The question** arises: do  $\hat{G}$ -movable chambers exist, or under what conditions?

The requested chambers are not always present. An important class of counterexamples is supplied by spherical subgroups  $\hat{G} \subset G$ , where  $\dim V_\lambda^{\hat{G}} \leq 1$ , there are no  $\hat{G}$ -movable line bundles and the quotient is a point. There are also non-spherical cases, like  $SL_2 \subset SL_2^{\times 4}$ , where the  $\hat{G}$ -movable cone is the diagonal ray

$\mathbb{R}_+\rho$ . In our previous work [ST15] we have obtained detailed results for  $\hat{G}$  a principal  $SL_2$ -subgroup of a semisimple group  $G$ ; in this case  $\hat{G}$ -movable chambers exist if  $\dim X \geq 5$ , which for simple  $G$  means not to be of type  $A_2$  or  $B_2$ .

Using our formula for the unstable locus, we obtain a concrete description of a system of nested cones in  $\Lambda_{\mathbb{R}}^+$  defined by codimension of the unstable locus, beginning with the  $\hat{G}$ -ample and the  $\hat{G}$ -movable cones. Let us denote  $\text{cob}(w) = \text{codim}_X \hat{G}Bx_w$ , we call this number the cobreadth of  $w$ . Let  $W_{q-\text{cob}}$  denote the set of elements of given cobreadth  $q$  and let  $W_{q-\text{cob},\min}$  denote the subset of minimal elements with respect to the opposite Bruhat order.

**Corollary:** *Under the hypothesis of Theorem I, the sets  $C_q^{\hat{G}}(X) = \{\lambda \in \Lambda_{\mathbb{R}}^+ : \text{codim}_X X^{us}(\lambda) \geq q\}$ , defined for  $q \geq 1$ , form a finite sequence of nested rational polyhedral cones given by*

$$C_q^{\hat{G}}(X) = \{\lambda \in \Lambda_{\mathbb{R}}^+ : \lambda(w^{-1}h_j) \leq 0, \forall j, \forall w \in W_{(q-1)-\text{cob},\min}\}.$$

*The  $\hat{G}$ -ample and -movable cones are obtained for  $q = 1$  and  $2$ , respectively. Furthermore:*

(i) *Since  $\text{cob}(w) = 0$  is equivalent to  $B^-x_w = \hat{B}^-x_w$  we have*

$$C^{\hat{G}}(X) = \{\lambda \in \Lambda_{\mathbb{R}}^+ : \lambda(w^{-1}h_j) \leq 0, \forall j, \forall w \in W : B^-x_w = \hat{B}^-x_w(\min)\}.$$

(ii)  *$\hat{G}$ -movable chambers exist if and only if the cone  $C_2^{\hat{G}}(X)$  has full dimension.*

(iii) *The subgroup  $\hat{G} \subset G$  spherical if and only if  $C_2^{\hat{G}}(X) = \{0\}$ , i.e.,  $\lambda = 0$  is the only dominant weight satisfying the inequalities  $\lambda(w^{-1}h_j) \leq 0$ , for  $w \in W_{1-\text{cob},\min}$  and  $j = 1, \dots, \ell$ .*

The combinatorics of Weyl groups arising around the dimension formula are also related to cohomology of nilpotent Lie subalgebras, via Kostant's theorem and the Belkale-Kumar product. We apply this relation in our setting, which yields certain faces on the boundary of  $C^{\hat{G}}(X)$ , called cohomological faces in the sense of [T13a], shown here to consist indeed of line bundles with unstable locus of codimension 1, see Theorem 6.27. For diagonal embeddings, using only rough estimates we are able to prove the following.

**Corollary:** *For a diagonal embedding  $\hat{G} \subset G = \hat{G}^{\times k}$  with sufficiently large  $k$  the following hold: (a) all GIT-classes whose closure intersects the regular boundary of  $C^{\hat{G}}(X)$  have unstable locus of codimension 1, and are not  $\hat{G}$ -movable; (b) for  $\lambda$  in a neighbourhood of  $\rho$  the codimension tends to infinity with the number of factors, thus  $\hat{G}$ -movable chambers exist.*

The second main result of this article concerns the quotients arising from  $\hat{G}$ -movable chambers, their Picard groups and Cox rings. Refining the aforementioned results [S14] on the effective cone on the quotient, we find a natural identification between the GIT-equivalence relation in  $\text{Pic}(X)$  with the Mori equivalence relation in  $\text{Pic}(Y)$ . The proofs appear in Theorems 7.4 and 7.3. The definition of Mori chambers can be found in Section 7, see also [HK00] for notions concerning Mori dream spaces.

**Theorem II:** *Suppose that there exists a  $\hat{G}$ -movable chamber  $C \subset C^{\hat{G}}(X)$  and let  $Y = Y_C$  be the corresponding GIT-quotient of  $X$ . Then  $Y$  is a Mori dream space and there is a canonical isomorphism  $\mathbb{R}$ -Picard groups giving rise to the following*

identifications:

$$\begin{array}{lll}
\mathrm{Pic}(X)_{\mathbb{R}} & \cong & \mathrm{Pic}(Y)_{\mathbb{R}} \\
C^G(X) & \cong & \overline{\mathrm{Eff}}(Y) \\
\text{GIT-chambers} & \leftrightarrow & \text{Mori chambers} \\
\mathrm{Mov}^{\hat{G}}(X) & \cong & \mathrm{Mov}(Y) \\
\overline{C} & \cong & \mathrm{Nef}(Y) \\
\mathrm{Cox}(X)^{\hat{G}} \cong & \bigoplus_{\lambda \in \Lambda^+} V_{\lambda}^{\hat{G}} \cong & \text{Finite extension of } \mathrm{Cox}(Y) .
\end{array}$$

Moreover, all rational contractions of  $Y$  to normal projective varieties are induced by VGIT from  $X$ .

Let us note that the family of Mori dream spaces produced as GIT-quotients of flag varieties could be of independent interest. For the sake of representation theory, clearly a concrete model for  $Y$  would be of great benefit – as explained above, this variety would encode the full information on dimensions of  $\hat{G}$ -invariants in  $G$ -modules for the given subgroup  $\hat{G} \subset G$ . Although we are able to prove many nice properties, these spaces remain somewhat implicit, as is often the case with quotients, due to the implicit nature of the fundamental existence results in invariant theory. The same is true to some extent for Mori dream spaces since several general constructions involve quotients, while many explicit alterations of varieties destroy the Mori dream property. It is therefore of interest to know whether our quotients appear among the known examples of Mori dream spaces. Perhaps the interaction of the Mori theory with the structure theory of semisimple groups could help obtain more concrete information about this family of spaces, ideally build concrete models at least for special classes of subgroups like diagonals.

### 3. THE HILBERT-MUMFORD CRITERION

Our approach to instability is based, as is often the case, on a fundamental result of Hilbert reducing instability for a reductive group to instability for its one-parameter subgroups, developed further by Mumford, cf. [FMK94] for the general theory and [N84] for a shorter presentation suitable for our purposes.

**Hilbert's theorem:** *Let  $H \rightarrow GL(V)$  be a representation of a reductive complex algebraic group. Then the invariant ring  $\mathbb{C}[V]^H$  is generated by a finite number of homogeneous elements. Let  $J \subset \mathbb{C}[V]^H$  be the vanishing ideal of 0 and let  $V_H^{us} \subset V$  denote its zero locus. Then*

$$V_H^{us} = \{v \in V : \overline{Hv} \ni 0\} = \{v \in V : \exists \gamma \in \mathrm{Hom}(\mathbb{C}^*, H) : \lim_{t \rightarrow 0} \gamma(t)v = 0\} .$$

The homogeneity of the generators ensures that  $\mathbb{P}(V)_H^{us}$  is well defined. For a projective variety  $Z \subset \mathbb{P}(V)$  preserved by  $H$ , the zero locus of  $J|_Z$  in  $Z$  is obtained by intersection  $Z_H^{us} = Z \cap \mathbb{P}(V)_H^{us}$ . The restriction of  $\mathcal{O}_{\mathbb{P}(V)}(1)$  to  $Z$  is an equivariant ample line bundle  $\mathcal{L}$ .

Mumford has devised a numerical criterion for instability for equivariant ample line bundles. We shall give a general statement, but in order to keep the notation simple we return to the case in hand,  $H = \hat{G}$ ,  $Z = X = G/B$ ,  $\mathcal{L} = \mathcal{L}_{\lambda}$ ,  $V = V_{\lambda}$ , with some  $\lambda \in \Lambda^{++}$  fixed for this section. This line bundle is ample, giving rise to a projective embedding obtained as the orbit of a highest weight vector:

$$X \cong G[v^{\lambda}] \subset \mathbb{P}(V_{\lambda}) .$$

We identify the elements  $\gamma \in \text{Hom}(\mathbb{C}^*, G)$  by their infinitesimal generators in the Lie algebra  $\xi = \dot{\gamma}(1) \in \mathfrak{g}$  and call them one-parameter subgroups (OPS). We consider the  $\xi$ -unstable locus taking the direction into account:

$$X_\xi^{us}(\lambda) = \{[v] \in X : \lim_{t \rightarrow -\infty} \exp(t\xi)v = 0\}.$$

Let us fix a pair of Cartan and Borel subgroups  $\hat{T} \subset \hat{B} \subset \hat{G}$ . The OPS of  $\hat{T}$  form a lattice naturally identified with the dual to the weight lattice  $\hat{\Gamma} = \hat{\Lambda}^\vee \subset \hat{\mathfrak{t}}$ . Recall that every OPS of  $\hat{G}$  is conjugate to a unique element of  $\hat{\Gamma}^+$ , the set of dominant elements with respect to  $\hat{B}$ .

Mumford's numerical function for  $\xi \in \hat{\Gamma}$ ,

$$(3) \quad M^\xi : X \rightarrow \mathbb{Z},$$

is defined as follows. For  $x \in X$  let  $x_0 = \lim_{t \rightarrow -\infty} \exp(t\xi)x \in X$ . The limit point belongs to the fixed set of the OPS,  $x_0 \in X^\xi$ . The connected components of  $X^\xi$  are contained in the projectivizations of the eigenspaces of  $\xi$ . Define  $M^\xi(x)$  to be the eigenvalue of  $\xi$  at  $x_0$ . The point is  $\xi$ -unstable if the eigenvalue is positive.

**Hilbert-Mumford criterion:** *Let  $\mathcal{L}$  be a  $\hat{G}$ -equivariant ample line bundle on  $X$ , in the above notation  $\mathcal{L}_\lambda$ ,  $\lambda \in \Lambda^{++}$ . A point  $x \in X$  is  $\hat{G}$ -unstable if and only if it is unstable for some one-parameter subgroup of  $\hat{G}$ . We have*

$$X_{\hat{G}}^{us}(\lambda) = \hat{G}X_{\hat{T}}^{us}(\lambda) = \hat{G} \left( \bigcup_{\xi \in \hat{\Gamma}^+} X_\xi^{us}(\lambda) \right), \quad X_\xi^{us}(\lambda) = \{x \in X : M^\xi(x) > 0\}.$$

To interpret the criterion in the specific situation of  $X = G/B$ , it is convenient to consider  $T$  and  $B$  containing  $\hat{T}$  and  $\hat{B}$ , respectively. Such extensions always exist, in general there are many, and this plays a role in our approach. The extensions of Borel subgroups correspond to the closed  $\hat{G}$ -orbits in  $X$ , which are all of the form  $\hat{G}/\hat{B}$ . The closed  $\hat{G}$ -orbits are all unstable for all  $\lambda^{++}$ . The extensions of Cartan subgroups allows to evaluate weights of  $T$  on elements of  $\hat{T}$ , and compute with the criterion. We obtain  $\iota : \hat{\Gamma} \subset \Gamma$  and  $\iota^* \Lambda \rightarrow \hat{\Lambda}^+$ . Notice that, for any nested pair of Borel subgroups we have  $\iota : \Lambda^+ \rightarrow \hat{\Lambda}^+$ , but  $\hat{\Gamma}^+$  is not necessarily contained in  $\Gamma^+$ . The hypothesis for our proof of the formula for the  $\hat{G}$ -unstable locus is that  $\hat{T}$  contains regular elements of  $G$  and hence determines a unique  $T$ . In this situation, the eigenvalues of any  $\xi \in \hat{\Gamma}$  in  $V_\lambda$ , and on  $X^\xi$  in particular, are the values on  $\xi$  of the  $T$ -weights. To this end we introduce the following notation. Let  $\mathcal{P}(\lambda) \subset \Lambda$  denote the set of  $T$ -weights of the irreducible  $G$ -module  $V_\lambda$ . For any  $x \in X$ , let  $\tilde{x} \in V_\lambda$  be a vector with  $[\tilde{x}] = x \in X \subseteq \mathbb{P}(V_\lambda)$ . We can decompose  $\tilde{x}$  as a sum of weight vectors,

$$(4) \quad \tilde{x} = \sum_{\mu \in \mathcal{P}(\lambda)} v_\mu.$$

For  $x \in X$ , let  $St(x) \subseteq \mathcal{P}(\lambda)$  denote the set of weights  $\mu$  for which  $v_\mu \neq 0$  in the decomposition (4). Then we have  $M^\xi(x) = \min\{\mu(\xi) : \mu \in St(x)\}$ .

#### 4. REGULAR ONE-PARAMETER SUBGROUPS

Recall that an element  $\xi \in \mathfrak{g}$  is called regular if its centralizer is a Cartan subalgebra. An elements of a fixed Cartan subalgebra  $\mathfrak{t}$  is regular, if no root in  $\Delta = \Delta(\mathfrak{g}, \mathfrak{t})$  vanishes on it. Thus a regular element also determines a Weyl chamber and a unique Borel subalgebra. The regular OPS of  $T$  correspond to the regular



points in the dual to the weight lattice  $\Gamma = \Lambda^\vee \subset \mathfrak{t}$ . We have  $\hat{\Gamma} \subset \Gamma$  as a sublattice. On  $\hat{\Gamma}$  (and on  $\hat{\mathfrak{t}}$  in general) we have two notions of regularity: with respect to  $G$  and  $\hat{G}$ . Since we have the inclusion  $\iota : \hat{\mathfrak{g}} \subset \mathfrak{g}$ , we have  $\iota^* \Delta \subset \hat{\Delta}$ , so  $G$ -regular implies  $\hat{G}$ -regular, but in general the converse implication does not hold. It holds if and only if any Weyl chamber  $\hat{\mathfrak{t}}_+$  is contained in a unique Weyl chamber  $\mathfrak{t}_+$ . We shall consider this hypothesis later on.

We focus first on  $G$ -regularity, and to simplify the notation we denote with subscript “reg” the  $G$ -regular elements and use a more detailed notation for other notions of regularity. So for instance  $\Gamma_{\text{reg}}$  and  $\hat{\Gamma}_{\text{reg}}$  denote the subsets of  $G$ -regular elements in  $\Gamma$  and  $\hat{\Gamma}$ , respectively. Also  $\mathfrak{t}_+^\circ$  denotes the interior of the Weyl chamber  $\mathfrak{t}_+$ . We record a few elementary but fundamental facts about regular elements and their action on the flag variety.

**Lemma 4.1.** *Let  $\xi \in \Gamma_{\text{reg}} \cap \mathfrak{t}_+$  be a regular dominant OPS.*

- (i) *The fixed points of  $\xi$  on  $X$  are the  $T$ -fixed points:  $X^\xi = X^T = \{x_w : w \in W\}$ .*
- (ii) *For every  $w \in W$  we have  $Bx_w = \{x \in X : \lim_{t \rightarrow -\infty} \exp(t\xi)x = x_w\}$ .*
- (iii) *For  $\lambda \in \Lambda^+$ , the Mumford function defined in (3) is given by*

$$M^\xi(x) = M^\xi(x_w) = w\lambda(\xi) \quad \text{for } x \in Bx_w.$$

*Proof.* The torus  $T$ , being the centralizer of  $\xi$ , acts on  $X^\xi$  which is a closed set. Hence, if  $X^\xi$  has a connect component of positive dimension, then this component contains  $T$ -stable curves. The  $T$ -stable curves in  $X = G/B$  are rational curves, exactly the closed orbits of the  $SL_2$ -subgroups of the roots. If such a curve belongs to  $X^\xi$ , then  $\xi$  belongs to the kernel of the corresponding root, which comes in contradiction with the regularity of  $\xi$ . Hence  $X^\xi$  is a finite set and consists of the  $T$ -fixed points, which proves part (i).

For part (ii), recall that the set of  $T$ -weights of the tangent space  $T_{x_w}Bx_w$  is exactly the inversion set  $\Phi_{w^{-1}} = \Delta^+ \cap w\Delta^-$ . Since the inversion set is closed under root addition, it gives rise to a subgroup  $N(w)$  contained in the unipotent radical  $N$  of  $B$ . The subgroup  $N(w)$  acts simply transitively on the Schubert cell,  $Bx_w = N(w)x_w$ . The inversion set is also the set of the positive eigenvalues of  $\xi$  in  $T_{x_w}X$ . The  $\xi$ -action on the Schubert cell  $N(w)x_w$  can be linearized using the fact that the exponential map of  $G$  restricted to  $\mathfrak{n}$  is an isomorphism of varieties onto its image  $N$ . Thus for  $x \in Bx_w$ , we have  $\lim_{t \rightarrow -\infty} \exp(t\xi)x = x_w$ . Since the Schubert cells constitute a cell decomposition of  $X$  and each cell adheres to its own  $\hat{T}$ -fixed point, we can deduce part (ii).

Recalling that the  $B$ -orbits  $Bx_w$  form the Schubert cell decomposition of  $X$ , part (iii) follows from (ii) and the fact that the map  $\phi_\lambda : X \rightarrow \mathbb{P}(V)$  defined by the line bundle  $\mathcal{L}_\lambda$  sends  $x_w$  to the extreme weight vector  $[v_{w\lambda}]$   $\square$

The Hilbert-Mumford criterion brings us to consider the following partition of the Weyl group determined by a pair in  $(\lambda, \xi) \in \Lambda \times \Gamma$ :

$$(5) \quad \begin{aligned} W &= W^+(\lambda, \xi) \sqcup W^0(\lambda, \xi) \sqcup W^-(\lambda, \xi) \quad , \\ W^+(\lambda, \xi) &= \{w \in W : w\lambda(\xi) > 0\} \quad , \\ W^0(\lambda, \xi) &= \{w \in W : w\lambda(\xi) = 0\} \quad , \\ W^-(\lambda, \xi) &= \{w \in W : w\lambda(\xi) < 0\} \quad . \end{aligned}$$

We have the following.

**Lemma 4.2.** *Let  $\lambda \in \Lambda^+$  and  $\xi \in \mathfrak{t}_+$  and put  $W^{\mp 0} = W^{\pm 0}(\lambda, \xi)$ . If  $w, w' \in W$  are related by the Bruhat order as  $w' \preceq w$ , then  $w'\lambda(\xi) \geq w\lambda(\xi)$  for all  $\xi \in \mathfrak{t}_+$ .*

*Consequently, if  $w$  belongs to either  $W^+$  or  $W^+ \cup W^0$ , then so do all elements smaller than  $w$ .*

*Proof.* The Bruhat order is defined by  $w' \preceq w$  if  $x_{w'} \in \overline{Bx_w} \subset X$  with  $w' \prec w$  if  $w' \neq w$ . The linear span of the Schubert variety in  $V_\lambda$  is the Demazure  $B$ -module  $V_{B,w\lambda}$  whose weights are exactly the weights of  $V_\lambda$  contained in  $w\lambda + Q_+$ . Thus  $w'\lambda = w\lambda + q$  for some sum of positive roots  $q$ . If  $h \in i\mathfrak{t}_+$ , then  $q(h) \geq 0$  and hence  $w'\lambda(h) \geq w\lambda(h)$ .  $\square$

**Corollary 4.3.** *For  $\lambda \in \Lambda^+$  and  $\xi \in \mathfrak{t}_+^\circ$ , we have*

$$X_\xi^{us}(\lambda) = \bigcup_{w \in W^+(\lambda, \xi)} Bx_w.$$

Let  $\xi_1, \xi_2$  be two regular dominant OPS, then

$$X_{\xi_1}^{us}(\lambda) = X_{\xi_2}^{us}(\lambda) \iff \Delta_{\xi_1}^+ = \Delta_{\xi_2}^+, \quad W^+(\lambda, \xi_1) = W^+(\lambda, \xi_2)$$

**Definition 4.4.** For  $x \in X$ , and a weight  $\mu \in \mathcal{P}(\lambda)$ , define the closed convex cone

$$C(x, \mu) := \{\xi \in \hat{\Gamma}_{\mathbb{R}} : \forall \nu \in St(x), \langle \mu, \xi \rangle \leq \langle \nu, \xi \rangle\}$$

of  $\Gamma_{\mathbb{R}}$ .

If  $\mathcal{P}(\lambda, x)$  denotes the set of weights  $\mu$  in  $\mathcal{P}(\lambda)$  for which  $C(x, \mu)_{reg} := C(x, \mu) \cap (\hat{\Gamma}_{\mathbb{R}})_{reg} \neq \emptyset$ , then the argument above shows that

$$(\hat{\Gamma}_{\mathbb{R}})_{reg} = \bigcup_{w\lambda \in \mathcal{P}(\lambda, x)} C(x, w\lambda)_{reg}.$$

Taking closures thus yields

$$(6) \quad \hat{\Gamma}_{\mathbb{R}} = \bigcup_{w\lambda \in \mathcal{P}(\lambda, x)} C(x, w\lambda).$$

We are now ready to reduce the  $\hat{T}$ -semistability for the line bundle  $\mathcal{L}_\lambda$  on  $X$  to  $\lambda$ -semistability for regular one-parameter subgroups.

**Proposition 4.5.** *Assume that  $\hat{G}$  contains regular one-parameter subgroups of  $G$ . Let  $\lambda \in \Lambda^{++}$  and  $x \in X$ .*

*If  $x$  is  $\mathcal{L}_\lambda$ -semistable for all regular one-parameter subgroups  $\xi$  of  $T$ , then  $x$  is  $\mathcal{L}_\lambda$ -semistable for all one-parameter subgroups of  $\hat{T}$ .*

*If  $x \in X$  is  $\mathcal{L}_\lambda$ -unstable for  $\hat{G}$ , then there exist a point  $y \in \hat{G}x$  and one-parameter subgroup  $\xi$  of  $\hat{T}$  which is regular in  $T$ , such that  $y$  is  $\mathcal{L}_\lambda$ -unstable for  $\xi$ .*

*Proof.* The two statements are clearly equivalent, so we prove the first one, but we record the second for further use. Let  $\xi \in \hat{\Gamma}$  be an arbitrary one-parameter subgroup of  $\hat{T}$ . Then there exists a  $w \in W$  such that  $w\lambda \in \mathcal{P}(\lambda, x)$  and  $\xi \in C(x, w\lambda)$ . We can thus write  $\xi$  as a limit  $\xi = \lim_{k \rightarrow \infty} \xi_k$ , where  $\{\xi_k\}_{k=1}^\infty$  is a sequence in  $C(x, w\lambda)_{reg}$  and the limit is with respect to the topology of the  $\mathbb{R}$ -vector space  $\hat{\Gamma}_{\mathbb{R}}$ . Then,

$$\langle w\lambda, \xi \rangle = \lim_{k \rightarrow \infty} \langle w\lambda, \xi_k \rangle \leq 0,$$

and hence  $x$  is  $\lambda$ -semistable for the one-parameter subgroup  $\xi$  by the Hilbert-Mumford criterion.  $\square$

**Corollary 4.6.** *With the assumption that  $\hat{G}_{reg} \neq \emptyset$ , For  $\lambda \in \Lambda^+$ , we have*

$$X_{\hat{G}}^{us}(\lambda) = \hat{G} X_{(\mathfrak{t}_+)_{reg}}^{us}(\lambda).$$

**4.1. Compatible Weyl chambers and cubicles.** Under the hypothesis that  $\hat{T}$  contains regular elements of  $T$ , the fixed Weyl chamber  $\hat{\mathfrak{t}}_+$  intersect the interior of some Weyl chambers of  $\mathfrak{t}$ . Following Berenstein and Sjamaar, we call two Weyl chambers  $\hat{\mathfrak{t}}_+$  and  $\mathfrak{t}_+$  *compatible* if  $\dim \hat{\mathfrak{t}}_+ \cap \mathfrak{t}_+ = \dim \hat{\mathfrak{t}}$ . Let us fix from now on a compatible pair of chambers  $\hat{\mathfrak{t}}_+$  and  $\mathfrak{t}_+$ . Then the Weyl chambers of  $\mathfrak{t}$  are parametrized by Weyl group elements, and the chambers compatible with  $\hat{\mathfrak{t}}_+$  determine the following set

$$W_{\text{com}} = \{w \in W : \dim(\hat{\mathfrak{t}}_+ \cap w\mathfrak{t}_+) = \dim \hat{\mathfrak{t}}\},$$

called *the compatible Weyl set*. Berenstein and Sjamaar observed that the Weyl group of the centralizer  $Z_K(\hat{T})$  acts on  $W_{\text{com}}$ , and defined *the relative Weyl set*  $W_{\text{rel}}$  to be the set of shortest representatives of the respective coset space. When  $\hat{K}$  contains regular elements of  $K$ , the centralizer  $Z_K(\hat{T})$  is a Cartan subgroup  $T$  of  $K$  and has trivial Weyl group  $W_T(T)$ . Hence with our assumption we have

$$W_{\text{com}} = W_{\text{rel}} = \{w \in W : \hat{\mathfrak{t}}_+^\circ \cap w\mathfrak{t}_+^\circ \neq \emptyset\}.$$

For  $\sigma \in W_{\text{com}}$ , the cone

$$\hat{\mathfrak{t}}_\sigma = \hat{\mathfrak{t}}_+ \cap \sigma\mathfrak{t}_+$$

is called a *cubicle* in  $\hat{\mathfrak{t}}$ . We have

$$\hat{\mathfrak{t}}_+ = \bigcup_{\sigma \in W_{\text{com}}} \hat{\mathfrak{t}}_\sigma.$$

Note that the Borel subgroups  $B^\sigma$  for  $\sigma \in W_{\text{com}}$  contain the fixed  $\hat{B}$ , but they might not be all Borel subgroups of  $G$  containing  $\hat{B}$ . More occur, for instance, for a root-subgroup  $SL_2 \subset SL_3$ .

Since  $T = Z_G(\hat{T}) = Z_G(T)$  and  $Z_G(\hat{T}) \subset N_G(\hat{T})$  is a normal subgroup, we have  $N_{\hat{K}}(\hat{T}) \subset N_K(\hat{T}) \subset N_K(T)$ . This yields an inclusion

$$j : \hat{W} \subset W.$$

The inclusion  $\hat{T} \subset T$  is equivariant with respect to  $j$ . There is a  $j$ -duality involution on  $W$  given by  $w \mapsto w^* = j(\hat{w}_0)ww_0$ . Lemma 2.4.3. in [BS00] states that  $W_{\text{com}}$  is stable under  $j$ -duality, and the cubicles are permuted by  $\hat{\mathfrak{t}}_{\sigma^*} = -\hat{w}_0\hat{\mathfrak{t}}_\sigma$  for  $\sigma \in W_{\text{com}}$ .

Let  $\sigma \in W_{\text{com}}$  and  $\lambda \in \Lambda^{++}$ . Then  $\iota^*(\sigma\lambda) \in \tilde{\Lambda}^+$  and hence  $\sigma \in W^+(\lambda, \hat{h})$  and  $j(\hat{w}_0)\sigma \in W^-(\lambda, \hat{h})$  for any  $\hat{h} \in \hat{\mathfrak{t}}_+^\circ$ . If  $\hat{h} \in \hat{\mathfrak{t}}_\sigma$ , then  $\sigma\lambda(\hat{h})$  is the maximum of  $W\lambda(\hat{h})$ .

## 5. A FORMULA FOR THE UNSTABLE LOCUS

As a consequence of the results of the previous section, we derive here our main formula for the unstable loci in  $X = G/B$  for the action of subgroup  $\hat{G} \subset G$  containing regular elements. Any regular one-parameter subgroup is contained in a unique cubicle  $\xi \in (\hat{\mathfrak{t}}_\sigma)_{\text{reg}}$ . For  $\lambda \in \Lambda^{++}$ , Corollary 4.3 implies that

$$X_\xi^{us}(\lambda) = \bigcup_{w \in W^+(\lambda, \sigma^{-1}\xi)} B^\sigma x_{\sigma w}.$$

Consider the set

$$W^{0-}(\lambda, \sigma) := \{w \in W : w\lambda(\sigma^{-1}(\hat{\mathfrak{t}}_\sigma^\circ)) \subset \mathbb{R}_{\leq 0}\} = W \setminus \left( \bigcup_{\xi \in \hat{\mathfrak{t}}_\sigma} W^+(\lambda, \sigma^{-1}\xi) \right).$$

**Remark 5.1.** Let  $\hat{\mathfrak{t}}_\sigma^\circ$  denote the relative interior of the cubicle in its real span  $\hat{\Gamma}_{\mathbb{R}}$ , we call it the open cubicle. The possible images of such an open cone under a real linear functional, like  $\sigma w\lambda$ , are the open cones in  $\mathbb{R}$  and the point zero:  $\mathbb{R}_+$  (attained at  $\lambda$ ),  $\{0\}$ ,  $\mathbb{R}$ ,  $\mathbb{R}_-$  (attained at  $\sigma w_0\lambda$ ). Thus, if  $w \in W^{0\pm}(\lambda, \sigma)$  and  $\sigma w\lambda(\hat{\mathfrak{t}}_\sigma^\circ) \ni 0$ , then  $w\lambda(\hat{\mathfrak{t}}_\sigma^\circ) = 0$  and so  $\iota^*w\lambda = 0$ . Therefore we have

$$W^{0\pm}(\lambda, \sigma) = W^\pm(\lambda, \sigma) \sqcup W^0(\lambda, \sigma) \quad , \quad W^0(\lambda, \sigma) = \{w \in W : \iota^*\sigma w\lambda = 0\}.$$

Note furthermore that

$$w_0 W^\pm(\lambda, \sigma) = W^\mp(\lambda, \sigma) .$$

**Theorem 5.2.** *Suppose that  $\hat{G}$  contains regular semisimple elements of  $G$ . Let  $\lambda \in \Lambda^{++}$ . Then the  $\hat{G}$ -unstable locus in  $X$  with respect to  $\mathcal{L}_\lambda$  is given by the  $\hat{G}$ -saturation of a union of Schubert cells for the Borel subgroups  $B^\sigma$  with  $\sigma \in W_{\text{com}}$*

$$X_{\hat{G}}^{us}(\lambda) = \bigcup_{(\sigma, w) \in W_{\text{com}} \times W : w \in W \setminus W^{0-}(\lambda, \sigma)} \hat{G} B^\sigma x_{\sigma w} .$$

The closure of  $\hat{G} B^\sigma x_{\sigma w}$  in  $X$  is an irreducible variety of dimension

$$\dim \hat{G} B^\sigma x_{\sigma w} = \dim B^\sigma x_{\sigma w} + \dim \hat{N}^- x_{\sigma w} = l(w) + \hat{n} - \#\hat{\Psi}_{\sigma w} ,$$

where  $\hat{n} = \dim \mathfrak{n}$  and  $\hat{\Psi} = \hat{\Delta}(\hat{\mathfrak{n}}^- \cap \mathfrak{b}^{\sigma w})$ . The codimension of the unstable locus is given by

$$\text{codim}_X X_{\hat{G}}^{us}(\lambda) = \min\{l(w_0 w) + \#\hat{\Psi}_{\sigma w} - \hat{n} : \sigma \in W_{\text{com}}, w \in W \setminus W^{0-}(\lambda, \sigma)\} .$$

*Proof.* From Corollary 4.6 we know that the  $\hat{G}$ -unstable locus is the  $\hat{G}$ -saturation of union of unstable loci for dominant one-parameter subgroups of  $\hat{T}$  regular in  $T$ . Let  $\xi \in (\hat{\Gamma}_+^{\text{reg}})$  be such a subgroup. Then  $\xi$  is contained in a unique cubicle  $\mathfrak{t}_\sigma$  with  $\sigma \in W_{\text{com}}$ . Thus  $\sigma^{-1}\xi$  is a regular integral element in  $\mathfrak{t}_+$  and we may apply Corollary 4.3. We get

$$X_{\sigma^{-1}\xi}^{us}(\lambda) = \bigcup_{w \in W^+(\lambda, \sigma^{-1}\xi)} B x_w .$$

Hence

$$X_\xi^{us}(\lambda) = \bigcup_{w \in W^+(\lambda, \xi)} B^\sigma x_{\sigma w} .$$

Now from Corollary 4.6 we get

$$X_{\hat{G}}^{us}(\lambda) = \hat{G} X_{(\mathfrak{t}_+^{\text{reg}})}^{us}(\lambda) = \hat{G} \left( \bigcup_{(\sigma, w) \in W_{\text{com}} \times W : w \in W \setminus W^{0-}(\lambda, \sigma)} B^\sigma x_{\sigma w} \right) .$$

This proves the formula for the unstable locus.

For the dimension formula, note that  $\hat{B} = \hat{G} \cap B^\sigma$  for every  $\sigma \in W_{\text{com}}$ . Hence

$$\dim \hat{G} B^\sigma x_{\sigma w} = \dim B^\sigma x_{\sigma w} + \dim \hat{N}^- x_{\sigma w} .$$

Since  $B^\sigma x_{\sigma w} = \sigma B x_w$ , we have  $\dim B^\sigma x_{\sigma w} = l(w)$ . The set  $\hat{\Psi}_{\sigma w}$  is by definition the set of weights of the  $\hat{N}^-$ -stabilizer of  $x_{\sigma w}$ . Thus we obtain  $\dim \hat{G} B^\sigma x_{\sigma w} = l(w) + \hat{n} - \#\hat{\Psi}_{\sigma w}$ . The codimension formula follows since  $n - l(w) = l(w_0 w)$ .  $\square$

**Remark 5.3.** The union in the formula for  $X_{\hat{G}}^{us}(\lambda)$  is not disjoint. For instance, note that  $W^{0-}(\lambda, \sigma)$  always contain the longest element  $w_0$ , so the union never runs over the whole Weyl group. However, for non- $\hat{G}$ -ample line bundles  $\mathcal{L}_\lambda$  we have  $X_{\hat{G}}^{us}(\lambda) = X$ , so all Schubert cells are unstable.

Since unstable loci are closed, we may consider their irreducible components. To this end, assuming for simplicity that we have a single cubicle, i.e.  $W_{\text{com}} = \{1\}$ , we have

$$X_{\hat{G}}^{us}(\lambda) = \bigcup_{w \in W \setminus W^{0-}(\lambda)} \overline{\hat{G} B x_w} .$$

Since  $B$  and  $\hat{G}$  are assumed to be connected, the closures  $\overline{\hat{G} B x_w}$  are irreducible. By leaving out closures included in other closures, i.e., when some inclusion  $\hat{G} B x_w \subseteq$

$\overline{\hat{G}Bx_{w'}}$  holds, we can write this union as an irredundant union over some subset  $I(\lambda) \subset W \setminus W^{0-}(\lambda)$ :

$$X_{\hat{G}}^{us}(\lambda) = \bigcup_{w \in I(\lambda)} \overline{\hat{G}Bx_w},$$

meaning that the  $\overline{\hat{G}Bx_w}$ , for  $w \in I(\lambda)$ , are the irreducible components of  $X_{\hat{G}}^{us}$ . Finding a good minimal set of representatives  $I(\lambda)$  would clearly help computations and understanding. It is thus of interest to understand exactly when inclusions of the form  $\overline{\hat{G}Bx_w} \subseteq \overline{\hat{G}Bx_{w'}}$ , with  $w, w' \in W \setminus W^{0-}(\lambda)$ , hold. In particular, when do we have equality?

## 6. THE CASE OF SINGLE CUBICLE

Here we make the hypothesis that all regular one-parameter subgroups of  $\hat{G}$  are also regular in  $G$ . In the terminology introduced above this assumption can be formulated in several equivalent ways. For instance: there is a single cubicle; the compatible Weyl set consists of the identity element; there is a unique embedding  $j : \hat{W} \subset W$  compatible with the restriction of weights; the interior of a given Weyl chamber  $\hat{t}_+$  intersects the interior of a single Weyl chamber  $t^+$ . Examples where this property is fulfilled are:

- diagonal embeddings  $\hat{G} \subset G = \hat{G}^{\times k}$ ;
- principal  $SL_2$ -subgroups  $SL_2 \cong \hat{G} \rightarrow G$ .
- subgroup containing principal  $SL_2$ -subgroups  $SL_2 \rightarrow S \subset \hat{G} \subset G$ .
- $SL_2 \subset SL_3$  given by any root.

In this setting, the formulae of Theorem 5.2 simplify and we obtain the following.

**Theorem 6.1.** *For  $\lambda \in \Lambda^{++}$  for the  $\hat{G}$ -unstable locus in  $X$  is given by the  $\hat{G}$ -saturation of a union of Schubert cells as follows:*

$$(7) \quad X_{\hat{G}}^{us}(\lambda) = \bigcup_{w \in W \setminus W^{0-}(\lambda)} \hat{G}Bx_w, \quad ,$$

where

$$\begin{aligned} W^{\pm}(\lambda) &= \{w \in W : w\lambda(\hat{t}_{\pm}^{\circ}) \subset \mathbb{R}_{\pm}\} \\ W^{0\pm}(\lambda) &= \{w \in W : w\lambda(\hat{t}_{\pm}^{\circ}) \subset \mathbb{R}_{\geq 0}\}. \end{aligned}$$

Denote  $n = \dim \mathfrak{n} = \dim X = \#\Delta^+$  and similarly  $\hat{n} = \dim \hat{\mathfrak{n}}$ . We have the following dimension formulae for the saturated Schubert cells and the unstable locus.

**Proposition 6.2.** *For  $w \in W$  denote*

$$\hat{\Psi}_w = \hat{\Delta}(\hat{\mathfrak{n}}^- \cap \mathfrak{b}^w) = \hat{\Delta}(\hat{\mathfrak{n}}^- \cap \mathfrak{n}_{\Phi_w}^w).$$

Then

$$\begin{aligned} \dim \hat{G}Bx_w &= \dim Bx_w + \dim \hat{N}^-x_w = l(w) + \hat{n} - \#\hat{\Psi}_w \\ \dim X - \dim \hat{G}Bx_w &= n - \hat{n} - l(w) + \#\hat{\Psi}_w = l(w_0w) - \hat{n} + \#\hat{\Psi}_w, \\ \dim X - \dim X_{\hat{G}}^{us}(\lambda) &= \min\{l(w_0w) - \hat{n} + \#\hat{\Psi}_w : w \in W \setminus W^{0-}(\lambda)\}. \end{aligned}$$

**Definition 6.3.** We define the breadth and cobreadth of a Weyl group element  $w \in W$  with respect to  $\hat{G}$ , as

$$\begin{aligned} b(w) &= \dim \hat{G}Bx_w = l(w) + \hat{n} - \#\hat{\Psi}_w, \\ \text{cob}(w) &= \text{codim} \hat{G}Bx_w = l(w_0w) + \#\hat{\Psi}_w - \hat{n}. \end{aligned}$$

We denote the subsets of  $W$  with a fixed breadth or cobreadth  $j$  by

$$W_{j-b} = \{w \in W : b(w) = j\} \quad , \quad W_{j-cob} = \{w \in W : cob(w) = j\} .$$

Also,  $W_{j-cob}(l)$  denotes the elements with length  $l$  and breadth  $j$ , and  $W_{j-cob, \min}$  denotes the set of minimal elements with respect to the Bruhat order in  $W_{j-cob}$ .

**Lemma 6.4.** *Let  $\lambda \in \Lambda^{++}$  and let  $w \in W$ .*

*If  $w \in W^{0+}(\lambda)$ , then  $\{w' \in W : w' < w\} \subset W^+(\lambda)$ .*

*If  $w \in W \setminus W^{0-}(\lambda)$ , then  $\{w' \in W : w' < w\} \subset W \setminus W^{0-}(\lambda)$ .*

*Proof.* The assumption that  $w' < w$  implies that  $w'\lambda \in w\lambda + \mathbb{N}\Delta^+$ , so that  $w'\lambda = w\lambda + q$ , where  $q$  is a sum of positive roots. If  $h_j$  is any simple coweight, we thus have

$$w'\lambda(h_j) = w\lambda(h_j) + q(h_j) \geq w(\lambda_j).$$

This proves both claims.  $\square$

**6.1. The  $\hat{G}$ -ample and movable cones.** We now proceed to explore the properties of the sets  $W^+(\lambda)$  and  $W^0(\lambda)$  for arbitrary  $\lambda \in \Lambda^+$ , and their reflections as properties of the unstable loci. Specifically we would like to estimate the codimension of  $N_\lambda$  in  $X$ . In particular, the  $\hat{G}$ -ample cone characterized by positive codimension of the unstable locus, and the  $\hat{G}$ -movable cone - by codimension strictly greater than one.

**Theorem 6.5.** *For  $\lambda \in \Lambda^{++}$ , we have*

$$\lambda \in C^{\hat{G}}(X) \text{ if and only if } W^{0-}(\lambda) \supset W_{0-cob, \min}.$$

and

$$\text{codim}_X X_G^{us}(\lambda) = \min\{j \in \mathbb{N} : W_{j-cob} \not\subset W^{0-}(\lambda)\} .$$

Consequently, the  $\hat{G}$ -ample cone is given by

$$C^{\hat{G}}(X) = \{\lambda \in \Lambda_{\mathbb{R}}^+ : \lambda(w^{-1}h_j) \leq 0, j = 1, \dots, \hat{\ell}, w \in W_{0-cob, \min}\} ,$$

where  $h_1, \dots, h_{\hat{\ell}}$  are the fundamental coweights of  $\hat{G}$ . The  $\hat{G}$ -movable cone is given by

$$\text{Mov}^{\hat{G}}(X) = \{\lambda \in \Lambda_{\mathbb{R}}^+ : \lambda(w^{-1}h_j) \leq 0, j = 1, \dots, \hat{\ell}, w \in W_{1-cob, \min}\} .$$

*Proof.* The theorem follows from our formula for the unstable locus, Theorem 6.1, and the codimension formula from Proposition 6.18. The  $\hat{G}$ -ample cone consists of the line bundles where the codimension is positive, so  $\lambda \in C^{\hat{G}}(X)$  is equivalent to  $W^{0-}(\lambda) \supset W_{0-cob}$ . Now notice that we may drop all nonminimal elements, because, by Lemma 6.4, if  $w \prec w'$  and  $w \in W^{0-}(\lambda)$  we have  $w' \in W^{0-}(\lambda)$ .  $\square$

**Corollary 6.6.** *If two elements of the  $\hat{G}$ -ample cone  $\lambda, \lambda' \in C^{\hat{G}}(X)$  define the same subsets of the Weyl group as  $W^{0-}(\lambda) = W^{0-}(\lambda')$ , then the line bundles  $\mathcal{L}_\lambda$  and  $\mathcal{L}_{\lambda'}$  are GIT-equivalent.*

**Remark 6.7.** The lists of inequalities given in the above theorem for the  $\hat{G}$ -ample and -movable cones may not be minimal. Namely, it is possible that  $w^{-1}h_j \in (\Lambda^-)^\vee$ , so that  $\lambda(w^{-1}h_j) \leq 0$  for all  $\lambda \in \Lambda^+$ .

**Example 1.** Let  $SL_2 \cong \hat{G} \subset G = SL_3$  be given by the  $SL_2$ -subgroup of the highest root with respect to some choice of Borel and Cartan subgroups on  $SL_3$ . Then  $W_{0-cob, \min} = W_{0-cob} = \{w_0\}$  and this element belongs to  $W^-(\lambda)$  for all  $\lambda \in \Lambda^+$ . Thus  $C^{\hat{G}}(X) = \Lambda_{\mathbb{R}}^+$ . Furthermore,  $W_{1-cob} = W \setminus \{1, w_0\} = W(1) \cup W(2)$ ,  $W_{1-cob, \min} = W(1)$ ,  $W_{2-cob} = \{1\}$ . We also have,  $s_\alpha \hat{h} = \beta^\vee \in (\Lambda^+)^\vee$ , where  $\{\alpha, \beta\} = \Pi$  are the simple roots of  $G$  and  $\hat{h}$  is the positive coroot of  $\hat{G}$ . Thus  $W_{1-cob, \min} \subset W^+(\lambda)$  for any  $\lambda \in \Lambda^{++}$  and hence  $\text{Mov}^{\hat{G}}(X) = 0$ .

**6.2. GIT-classes for  $\hat{T}$  and  $\hat{G}$ .** Here we compare the to the GIT-classes on  $X$  with respect to  $\hat{G}$  to those with respect to the maximal torus  $\hat{T} \subset \hat{G}$ . We begin with the following result, ensuring that the phenomenon of the so-called thick walls, [R98], does not occur for  $\hat{G}$ .

**Theorem 6.8.** ([S14]) *If a semisimple group  $\hat{G} \subset G$  acts on  $X = G/B$ , the GIT classes having full dimension in  $\text{Pic}(X)_{\mathbb{R}}$  are chambers.*

Denote  $\hat{\ell} = \dim \hat{T} = \text{rank}(\hat{G})$  and let  $h_1, \dots, h_{\hat{\ell}} \in \hat{\mathfrak{t}}_+ \subset \mathfrak{t}_+$  be the fundamental coweights of  $\hat{G}$ . Consider the hyperplanes in  $\Lambda_{\mathbb{R}}$  given by:

$$\mathcal{H}_{wh_j} = \{\xi \in \Lambda_{\mathbb{R}} : (\xi|wh_j) = 0\}, \quad w \in W, j \in \{1, \dots, \hat{\ell}\}.$$

This system of hyperplanes defines a fan  $\mathcal{F}$  in  $\Lambda_{\mathbb{R}}$  and we focus on the intersection  $\mathcal{F}^+$  of this fan with the positive Weyl chamber  $\Lambda_{\mathbb{R}}^+$ , in relation to the partition of the latter into GIT-equivalence classes. The cones of the fan  $\mathcal{F}$ , or rather their relative interiors, are determined by the following signature.

**Definition 6.9.** Define the  $\iota$ -signature of  $\lambda$  as

$$\begin{aligned} \iota\text{-sign} : \Lambda_{\mathbb{R}} &\rightarrow \{1, 0, -1\}^{\times \hat{\ell}|W|} \\ \iota\text{-sign}(\lambda) &= (\text{sign}(w\lambda(h_j))) ; w \in W, 1 \leq j \leq \hat{\ell}. \end{aligned}$$

Given  $\lambda \in \Lambda^+$ , its position with respect to the system of hyperplanes is determined by the system of  $\hat{\ell}$  partitions  $W = W^+(\lambda, h_j) \sqcup W^0(\lambda, h_j) \sqcup W^-(\lambda, h_j)$ ,  $j = 1, \dots, \hat{\ell}$ , defined previously. We have

$$W^{0\pm}(\lambda) = \bigcap_{j=1}^{\hat{\ell}} W^{0\pm}(\lambda, h_j).$$

Note that all  $\lambda$  belonging to a given cone  $F \in \mathcal{F}^+$  define the same sets  $W^{0\pm}(\lambda)$ , so it makes sense to denote  $W^{0\pm}(F) = W^{0\pm}(\lambda)$ .

As a corollary of Theorem 6.1 we get the following.

**Theorem 6.10.** *The  $\hat{T}$ -unstable locus in  $X$  with respect to  $\lambda \in \Lambda^{++}$  is given by*

$$X_T^{us}(\lambda) = \bigcup_{\hat{w} \in W} \bigcup_{w \in W \setminus W^{0-}(\lambda)} j(\hat{w})Bx_w.$$

*Each  $\hat{T}$ -GIT-class of ample line bundles on  $X$  is the union of the cones  $F$  in the fan  $\mathcal{F}^+$  yielding a given subset of  $W^{0+}(F) \subset W$  for  $\lambda$  in the GIT-class.*

*Proof.* We know that, if two dominant weights  $\lambda, \lambda' \in \Lambda^{++} \cap C^{\hat{G}}(X)$  have the same signature  $\iota\text{-sign}(\lambda) = \iota\text{-sign}(\lambda')$ , then they are  $\xi$ -GIT-equivalent for each dominant OPS  $\xi$ . Now the theorem follows from the fact that the Weyl chamber is a fundamental domain for the  $\hat{W}$ -action on  $\hat{\mathfrak{t}}_{\mathbb{R}}$ , equivariantly embedded in  $\mathfrak{t}$  with respect to the inclusion  $j : \hat{W} \rightarrow W$ .  $\square$

In order to understand how the  $\hat{T}$ -GIT-classes are built out the the cones of the fan  $\mathcal{F}$ , it would be useful to know how  $W^{0-}$  varies along the boundary of a given cone. If  $C, F \in \mathcal{F}^+$  are two cones such that  $F$  is a facet of  $C$ , and both intersect the interior of  $\mathfrak{t}_+$ , then  $F$  is obtained from  $\overline{C}$  by intersection with one hyperplane  $\mathcal{H}_{wh_j}$ . Notice that there could be more than one pair  $w, h_j$  defining the same hyperplane (in fact there are at least two giving the two opposite normals). Between  $C$  and  $F$  the sign  $\text{sign}(\lambda(wh_j))$  differs exactly at these pairs.

**Lemma 6.11.** *Let  $W_j = W_{h_j}$  denote the stabilizer of  $h_j$  in  $W$ . Let  $w, w' \in W$  and  $j, j' \in \{1, \dots, \hat{\ell}\}$ . Then*

- (1)  $wh_j = w'h_{j'}$  if and only if  $j = j'$  and  $w \in w'W_j$ ;
- (2)  $wh_j = -w'h_{j'}$  if and only if  $h_{j'} = h_j^* = -w_0h_j$  and  $w' \in wW_jw_0$ ;

Consequently, a given hyperplane  $\mathcal{H}_{wh_j}$ , with a chosen orientation of the normal, determines  $j$  uniquely and  $w$  up to its coset  $wW_j$ . The representatives of minimal or maximal length in this coset are unique. The opposite orientation corresponds to  $j^*$ . and  $ww_0W_{j^*}$ .

*Proof.* The statements follows immediately from the definitions and facts that  $h_1, \dots, h_\ell$  are dominant and linearly independent, and permuted by the dualization  $h_{j^*} = h_j^* = -w_0h_j$ .  $\square$

**Lemma 6.12.** *Let  $\hat{\mathfrak{g}} \subset \mathfrak{g}$  be an embedding of complex semisimple Lie algebras. Let  $\xi \in \hat{\mathfrak{g}}$  be a semisimple element and  $\hat{\mathfrak{g}}'_\xi, \mathfrak{g}'_\xi$  be the semisimple parts of its centralizers in the two algebras, respectively.*

- (1) *If  $\hat{\mathfrak{g}}$  contains regular semisimple elements of  $\mathfrak{g}$ , then  $\hat{\mathfrak{g}}'_\xi$  contains regular semisimple elements of  $\mathfrak{g}'_\xi$ .*
- (2) *If any Weyl chamber of  $\hat{\mathfrak{g}}$  is contained in a unique Weyl chamber of  $\mathfrak{g}$ , then the same property holds of the embedding  $\hat{\mathfrak{g}}'_\xi \subset \mathfrak{g}'_\xi$  as well.*

*Proof.* The lemma follows by direct verifications using the fact that centralizer subalgebras in semisimple Lie algebras are root-subalgebras, i.e., the root systems are subsystems of the root systems of  $\hat{\mathfrak{g}}$  and  $\mathfrak{g}$ , respectively.  $\square$

**Proposition 6.13.** *Suppose that  $C_1, C_2$  are two full-dimensional cones of the fan  $\mathcal{F}^+$  sharing a facet  $C_{12} = \overline{C}_1 \cap \overline{C}_2$ . Consider the following two sets consisting of Weyl group elements yielding normals to  $C_{12}$  with signs positive on  $C_1$  and  $C_2$ , respectively:*

$$W^k = \{w \in W : \exists j \in \{1, \dots, \hat{\ell}\} : C_{12}(w^{-1}h_j) = 0, C_k(w^{-1}h_j) > 0\} \quad , \quad k = 1, 2.$$

*Then there exists a unique  $h_{j_1}$  such that, upon setting  $h_{j_2} = -w_0h_{j_1}$ , the set  $W^k$  is a coset  $W_{j_k}w_k$  in  $W_{j_k}$ . Moreover,  $w_0W^1 = W^2$ . The elements  $w_k$  are determined uniquely by the additional requirement to have minimal length in the respective cosets. We have*

$$(8) \quad W^-(C_1) \cup W^-(C_2) = (W^-(C_1) \cap W^-(C_2)) \sqcup (W^-(C_1) \cap W_2) \sqcup (W^-(C_2) \cap W_1).$$

*Suppose furthermore that both  $C_1$  and  $C_2$  give rise to  $\hat{G}$ -ample GIT-chambers  $C_1, C_2$  in  $C^{\mathcal{G}}(X)$ . Then*

$$\begin{aligned} X^{us}(C_1) \cup X^{us}(C_2) &= X^{us}(C_1) \cup \left( \bigcup_{w \in W_2 \cap W^-(C_1)} \hat{G}Bx_w \right) \\ &= \bigcup_{w \in (W \setminus W^-(C_1)) \cup W_2} \hat{G}Bx_w \quad . \end{aligned}$$

*The above formula also holds if 1, 2 are switched.*

*Proof.* Since both unstable loci consist of  $\hat{G}$ -saturated Schubert cells, the GIT-classes differ if and only if some Schubert cell is unstable for one of the classes, but not for the other. The assumptions imply that  $C_1$  and  $C_2$  lie on two opposite sides of a hyperplane  $\mathcal{H}_{w^{-1}h_j}$ , with  $w$  in either  $W_1$  or  $W_2$  and some corresponding  $j \in \{1, \dots, \hat{\ell}\}$ . Thus the signatures  $\iota\text{-sign}(C_1)$  and  $\iota\text{-sign}(C_2)$  differ exactly at such pairs  $w, h_j$  and nowhere else. Say  $w \in W_1$ , so that  $C_1(w^{-1}h_j) > 0$  and  $C_2(w^{-1}h_j) < 0$ . Then  $w \in W \setminus W^{0-}(C_1)$  and hence  $\hat{G}Bx_w \subset X^{us}(C_1)$ . Thus  $W^k \subset W \setminus W^-(C_k)$  and  $W^-(C_1) \cup W^-(C_2) \subset W^-(C_k) \sqcup W_k$  for both  $k = 1, 2$ . This implies formula (8). Furthermore,  $\hat{G}Bx_w$  is not  $C_2$ -unstable if and only if  $w \in W^-(C_2)$ . The formula for the unstable locus is an immediate consequence in view of Theorem 6.1.  $\square$



**Corollary 6.14.** *With the assumptions and notation of the above proposition, the following are equivalent:*

- (i) *the cones  $C_1$  and  $C_2$  belong to distinct  $\hat{T}$ -GIT-classes;*
- (ii) *at least one of the sets  $W^1 \cap W^-(C_2)$  or  $W^2 \cap W^-(C_1)$  is nonempty;*
- (iii)  *$w_1 \in W^+(C_2)$ .*

**Theorem 6.15.** *Let  $C$  be a full-dimensional cone in the fan  $\mathcal{F}$  defining a  $\hat{G}$ -ample GIT-class. Let  $F \subset C$  be a regular facet of  $C$ . Let  $w_1, h_1$  be the unique pair, given by Proposition 6.13, such that*

$$W^1 = W_1 w_1 = \{w \in W : \exists j \in \{1, \dots, \hat{\ell}\} : F(w^{-1}h_1) = 0, C(w^{-1}h_j) > 0\}$$

*and  $w_1$  is of minimal length in  $W^1$ . Let  $G_1$  and  $\hat{G}_1$  be the semisimple parts of the centralizers, in  $G$  and  $\hat{G}$  respectively, of  $h_1$ . Then  $X_1 = G_1 x_{w_1}$  is a closed  $G_1$ -orbit in  $X$ ; the restriction  $F_1 = (w_1 F)|_{t_1}$  is a cone in the ample cone of  $X_1$ . Furthermore,  $F_1$  is a cone in the signature fan  $\mathcal{F}_1^+$  on  $X_1$  respect to the  $\hat{G}_1$ -action. Then*

$$W^{0-}(F) = W^{0-}(C) \sqcup W_1^{0-}(F_1)w_1.$$

*Consequently,  $F$  and  $C$  define distinct  $\hat{G}$ -GIT-classes if and only if  $F_1$  intersects, and hence belongs to, the  $\hat{G}_1$ -ample cone  $C^{\hat{G}_1}(X_1)$ .*

*Proof.* The statement follows from the above corollary and lemma.  $\square$

**Remark 6.16.** The above theorem yields a recursive algorithm for computation of the  $\hat{G}$ -GIT-chambers in  $\text{Pic}(X)$ . It consists of the following steps (with the notation of the proposition):

- 1) determine the signature fan  $\mathcal{F}$ ;
- 2) determine the  $\hat{T}$ -GIT-classes, as unions of cones  $F \in \mathcal{F}$  having the same set  $W^{0-}(F)$ ;
- 3) for each pair of neighbouring  $\hat{T}$ -classes of full dimension, pick any cone  $F \in \mathcal{F}$  defining the separating  $\hat{T}$ -wall, and check whether  $F_1$  is  $\hat{G}_1$ -ample on  $X_1$ , for instance, by checking whether  $W_1^{0-}(F_1)$  contains elements of cobreadth 0 in  $W_1$ . If  $F_1$  is  $\hat{G}_1$ -ample, then  $F$  also defines a  $\hat{G}$ -wall and the two  $\hat{T}$ -chambers are define distinct  $\hat{G}$ -chambers.

It is conceivable that this procedure can be generalized without great difficulties to determine all GIT-classes, rather than just the chambers. We confine ourselves to chambers since this is sufficient for our present purposes.

**Example 2.** The following example shows that the  $\hat{G}$ -classes and do not coincide with the  $\hat{T}$ -classes. In the notation of Theorem 6.15, this amounts to finding an  $F$  such that  $F_1$  is not  $\hat{G}_1$ -ample.

Consider the diagonal embedding  $\hat{G} = SL_3 \hookrightarrow SL_3^{\times 3} = G$ . Let  $\lambda = (\hat{\lambda}, \hat{\lambda} + \hat{\lambda}^*, \lambda^*)$ , with any  $\hat{\lambda} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$  satisfying  $a_1 > a_2 > 0$ . Then we have  $\lambda \in C^{\hat{G}}(X) \cap \mathcal{H}_{w_1^{-1}h_1}$ , where  $w_1 = (s_1 s_2, s_1, 1)$ . The semisimple centralizer subgroups are then given by  $\hat{G}_1 = SL_2 \hookrightarrow SL_2^{\times 3} = G_1$ , and the variety  $X_1$  is a triple product  $(\mathbb{P}^1)^{\times 3}$ . The element  $w_1$  is of minimal length in its coset by  $W_1 = \{1, s_2\}^{\times 3}$ . Further, we calculate that

$$\lambda_1 = w_1 \lambda|_{t_1} = \frac{1}{3}(a_1 - a_2, 3a_1 + 3a_2, 2a_1 + a_2).$$

The middle coordinate of this weight,  $3a_1 + 3a_2$ , exceeds the sum of the other two coordinates, which is  $3a_1$ . Hence, from our knowledge of the  $SL_2$ -ample cone for diagonal embeddings, we deduce that  $\lambda_1 \notin C^{\hat{G}_1}(X_1)$ .

### 6.3. Some general bounds on the codimension of the unstable locus.

**Remark 6.17.** 1) Notice the inequalities

$$\hat{n} - l(w_0 w) \leq \#\hat{\Psi}_w \leq \min\{l(w), \hat{n}\}.$$

2) We have

$$\begin{aligned} \text{codim}_X \hat{G}Bx_{w_0 w} = 0 &\iff l(w) + \#\hat{\Psi}_{w_0 w} = \hat{n} \\ &\implies l(w) \leq \hat{n} \end{aligned}$$

3) We have  $\lambda \in \Lambda^{++} \setminus C^G(X)$  if and only if there exists  $w \in W \setminus W^{0+}(\lambda)$  such that  $l(w) + \#\hat{\Psi}_{w_0 w} = \hat{n}$ .

**Proposition 6.18.** *Let  $\lambda \in \Lambda^{++}$  and  $l_\lambda = \max\{l \in \mathbb{N} : W(l) \subset W^{0+}(\lambda)\}$ . Then*

$$\text{codim}_X X_{\hat{G}}^{us}(\lambda) > l_\lambda - \hat{n}.$$

*In particular, if  $\lambda \notin C^{\hat{G}}(X)$ , i.e.,  $\text{codim}_X X_{\hat{G}}^{us}(\lambda) = 0$ , then  $l_\lambda < \hat{n}$ .*

*Proof.* Suppose first that the codimension is 0. We apply 2) from the above remark to get the existence of  $w \in W \setminus W^{0+}(\lambda)$  with  $l(w) \leq \hat{n}$ . Note that  $W(l) \subset W^{0+}(\lambda)$  implies  $W(l') \subset W^{0+}(\lambda)$  for all  $l' \leq l$ . Thus there exists  $w' \in W \setminus W^{0+}(\lambda)$  with  $l(w') = \hat{n}$  and hence  $l_\lambda < \hat{n}$ .

Now we proceed to prove the general inequality. Let  $w \in W$  be such that  $\hat{G}Bx_w \subset X_{\hat{G}}^{us}(\lambda)$ . Then  $w_0 w \notin W^{0+}(\lambda)$ . This implies  $l(w_0 w) > l_\lambda$  and from Proposition 6.2 we get

$$\text{codim}_X \hat{G}Bx_w \geq l(w_0 w) - \hat{n} > l_\lambda - \hat{n}.$$

This proves the proposition.  $\square$

**Remark 6.19.** The situation  $l_\lambda < n$  may occur for  $\lambda \in C^{\hat{G}}(X)$ . This happens, for instance, for a diagonal embedding in a two-fold product,  $\hat{G} \subset \hat{G} \times \hat{G}$ , with the property  $\hat{w}_0 \neq -1$ .

Let us observe how the number  $l_\lambda$  may vary as  $\lambda$  varies in  $\Lambda^{++}$ . It is clearly bounded; let  $l_{\max}$  be the maximal attained value. The condition  $W(l) \subset W^{0+}(\lambda)$  becomes stricter as  $l$  grows. We have a family of cones in  $\Lambda^+$  given by

$$\Lambda_l^+ = \{\lambda \in \Lambda^+ : l_\lambda \geq l\}, \quad \Lambda_{l_{\max}}^+ \subset \dots \subset \Lambda_{\hat{n}}^+ \subset \Lambda^+.$$

Note that, as a direct consequence of Theorem 6.5 and Proposition 6.18. we have

$$\Lambda_{\hat{n}}^+ \subset C^{\hat{G}}(X)$$

and

$$\Lambda_{\hat{n}+1}^+ \subset \text{Mov}^{\hat{G}}(X).$$

**Remark 6.20.** The above observation has the following consequence: if  $\Lambda_{\hat{n}+1}^+$  spans a full dimensional cone in  $\Lambda_{\mathbb{R}}$ , then there exist  $\hat{G}$ -movable GIT-chambers in the Picard group of  $X$ .

The following lemma describes a useful semicontinuity property for the sets  $W^+(\lambda)$  in dependence of  $\lambda$ .

**Lemma 6.21.** *For every  $\lambda \in \Lambda^{++}$  there exists an  $\varepsilon > 0$  such that, for  $\nu \in \Lambda_{\mathbb{R}}$  with  $\|\nu\| < \varepsilon$ , the inclusion  $W^\pm(\lambda) \subset W^\pm(\lambda + \nu)$  holds.*

*Proof.* The proof follows directly from the definition of  $W^\pm(\lambda)$  since the pairing between  $\Lambda_{\mathbb{R}}$  and  $\mathfrak{t}$  is continuous.  $\square$

**Proposition 6.22.** *Suppose that there exists  $\lambda \in \Lambda^{++}$  such that  $W^-(\lambda) \supset W_{1-\text{cob}, \min}$ , then there exist  $\hat{G}$ -movable chambers. In particular,  $\hat{G}$ -movable chambers exist, if there exists  $\lambda \in \Lambda^{++}$  with  $W^+(\lambda) \supset W(\hat{n} + 1)$ .*

*Proof.* The proposition follows directly from the above lemma.  $\square$

**Example 3.** (An estimate for the case  $\lambda = \rho$ )

Let  $\hat{G} \subset G = \hat{G}^{\times k}$  be the diagonal subgroup. Consider the smallest strictly dominant weight

$$\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha = (\hat{\rho}, \dots, \hat{\rho})$$

and the corresponding unstable locus  $X_{\hat{G}}^{us}(\rho)$ . Let  $w = (w_1, \dots, w_k) \in W$ . Then, for  $h \in \hat{\mathfrak{t}}$ , we have

$$w\rho(h) = \sum_{j=1}^k w_j \hat{\rho}(h) = k\hat{\rho}(h) - \sum_{j=1}^k \langle \Phi_{w_j} \rangle(h) .$$

Recall that  $l(w) = \sum_{j=1}^k \# \Phi_{w_j}$ . Let  $b_h = \max(\Delta(h))$  be the maximal value of a root on the fixed Cartan element. Then

$$w\rho(h) \geq k\hat{\rho}(h) - b_h \sum_{j=1}^k \# \Phi_{w_j} = k\hat{\rho}(h) - b_h l(w) .$$

This, along with Proposition 6.18, yields the following rough estimates.

- (i) If  $k > 2l(w)$ , then  $w \in W^+(\rho)$ .
- (ii) If  $k = q + 2\hat{n}$  with  $q \in \mathbb{N}$ , then

$$\text{codim}_X X_{\hat{G}}^{us}(\rho) \geq q/2 .$$

Hence, by Proposition 6.22 movable chambers exist for diagonal embedding with sufficiently many factors.

**Remark 6.23.** In our previous work, [ST15], we have considered the case where  $\hat{G}$  is a principal  $SL_2$ -subgroup  $G$ . We have shown that  $\hat{G}$ -movable chambers exist, except for a small number of degenerate cases for  $G$ . Under some more assumptions, e.g.  $G$  not having simple factors of rank 1 or 2, the entire ample cone is  $\hat{G}$ -movable.

**6.4. Cohomological components and Lie algebra cohomology.** The nilpotent Lie algebras  $\mathfrak{n}$  and  $\hat{\mathfrak{n}}$  have already appeared, with an interference of the Weyl group, in the dimension formula for the unstable locus, cf. theorems 5.2 and 6.1. These Lie algebras and their cohomology appear also in key places in previous works on the present topics. Recall that  $\Lambda \mathfrak{n}^*$  is the cochain complex for the Lie algebra cohomology of  $\mathfrak{n}$  with respect to the standard differential. The cohomology is computed in a theorem of Kostant, [Kos61], stated below. The cup product in  $H^*(\mathfrak{n})$  is isomorphic to the Belkale-Kumar product in  $H^*(X)$ , designed indeed for the minimization of the list of inequalities for  $C^{\hat{G}}(X)$  as a deformation of the usual cup product in  $H^*(X)$ , [BK06]. Thus the two cohomology spaces are isomorphic as vector spaces, with two ring structures related by a deformation. For complete flag varieties the two structures are indeed different unless  $X$  is a product of copies of  $\mathbb{P}^1$ . The corresponding pullback related to the embedding  $\hat{G} \subset G$  also differ and are related by a deformation, [RR11]. There is a relation between the pullback  $H^*(\mathfrak{n}) \rightarrow H^*(\hat{\mathfrak{n}})$  and certain cohomological faces of the  $\hat{G}$ -ample cone  $C^{\hat{G}}(X)$  has been established in [T13a]. In this section we show how these cohomological faces fit into the present context.

The basis  $\{e_\alpha : \alpha \in \Delta^+\}$  of  $\mathfrak{n}$  extends to a basis of decomposable tensors in  $\Lambda \mathfrak{n}$ :

$$\Lambda \mathfrak{n} = \text{span}\{e_\Phi : \Phi \subset \Delta^+\} \quad , \quad \text{where} \quad e_\Phi = \wedge_{\alpha \in \Phi} e_\alpha .$$

We denote by  $e_\Phi^* \in \Lambda \mathfrak{n}^*$  the element dual to  $e_\Phi$ . For  $w \in W$ , we denote  $e_w = e_{\Phi_w}$ . Thus  $e_{w_0}$  spans the top-degree component  $\Lambda^n \mathfrak{n}$ , and similarly  $\tilde{e}_{\tilde{w}_0}$  spans  $\Lambda^{\hat{n}} \hat{\mathfrak{n}}$ .

Kostant's theorem implies that the elements  $e_w^*$  are harmonic cocycles with respect a suitable Hermitian form related to the fixed maximal compact subgroup  $K$ . They form a  $T$ -weight basis of the space harmonics, and the weight space decomposition of  $H^\bullet(\mathfrak{n})$  is multiplicity free, given by

$$(9) \quad H^q(\mathfrak{n}) = \bigoplus_{w \in W(q)} H^\bullet(\mathfrak{n})^{-\langle \Phi_w \rangle} \quad , \quad H^\bullet(\mathfrak{n})^{-\langle \Phi_w \rangle} = \mathbb{C}[e_w^*] .$$

Under our current assumptions, we have an embedding of maximal nilpotent subalgebras  $\hat{\mathfrak{n}} \subset \mathfrak{n}$  induced by the embedding  $\iota : \hat{\mathfrak{g}} \subset \mathfrak{g}$ . This extends to an embedding of Grassmann algebras  $\iota : \Lambda \hat{\mathfrak{n}} \hookrightarrow \Lambda \mathfrak{n}$  and we have the resulting dual map  $\iota^* : \Lambda \mathfrak{n}^* \rightarrow \Lambda \hat{\mathfrak{n}}^*$ . We consider the image

$$\iota(\Lambda^{\hat{n}} \mathfrak{n}) \subset \Lambda^{\hat{n}} \mathfrak{n} .$$

It is generated by the element  $\iota \tilde{e}_{\tilde{w}_0}$ , which can be written in the above basis as

$$\iota \tilde{e}_{\tilde{w}_0} = \sum_{\Phi \in \Delta^+} a_\Phi^0 e_\Phi$$

**Definition 6.24.** We call a Weyl group element  $w \in W$  fit for  $\hat{G}$ , or just fit when  $\hat{G}$  is fixed, if  $l(w) = \hat{n}$  and  $\text{codim}_X \hat{G} B x_{w_0 w} = 0$ .

**Lemma 6.25.** *Let  $w \in W$ . The following are equivalent:*

- (i) *the element  $w$  is fit;*
- (ii)  $\iota^* e_{w^{-1}}^* = a_{w^{-1}} \tilde{e}_{\tilde{w}_0}^* \quad , \quad a_{w^{-1}} \neq 0$ ;
- (iii)  $\iota^* \langle \Phi_{w^{-1}} \rangle = 2\hat{\rho}$  and there is a bijective pullback in cohomology of line bundles along the embedding  $\varphi : \hat{X} \hookrightarrow X$ ,

$$\varphi_{w \cdot 0}^* : \mathbb{C} = H^{\hat{n}}(X, \mathcal{L}_{w \cdot 0}) \rightarrow H^{\hat{n}}(\hat{X}, \hat{\mathcal{K}}) = \mathbb{C} ,$$

where  $\mathcal{K} = \hat{\mathcal{L}}_{-2\hat{\rho}}$  is the canonical bundle on  $\hat{X}$ .

*Proof.* Recall the codimension formula  $\text{codim}_X \hat{G} B x_{w_0 w} = l(w) - \hat{n} + \# \hat{\Psi}_{w_0 w}$ , where  $\hat{\Psi}_w = \hat{\Delta}(\hat{\mathfrak{n}}^- \cap \mathfrak{b}^w)$ . Thus for  $w \in W(\hat{n})$  being fit is equivalent to  $\hat{\Psi}_{w_0 w} = \emptyset$  and also to

$$0 = \hat{\mathfrak{n}} \cap \mathfrak{b}^w = \hat{\mathfrak{n}} \cap \mathfrak{n}_{\Phi_{w_0 w}}^w .$$

Since  $w(\Phi_{w_0 w}) = \Phi_{w_0 w^{-1}}$ , we have  $\Delta^+ = w(\Phi_{w_0 w}) \sqcup \Phi_{w^{-1}}$  and  $\mathfrak{n} = \mathfrak{n}_{\Phi_{w_0 w}}^w \oplus \mathfrak{n}_{\Phi_{w^{-1}}}$ . Also,  $\dim \mathfrak{n}_{\Phi_{w^{-1}}} = l(w^{-1}) = l(w)$  and  $\Lambda^{l(w)} \mathfrak{n}_{\Phi_{w^{-1}}} = \mathbb{C} e_{w^{-1}}$ . It follows that,  $w$  is fit if and only if  $a_{\Phi_{w^{-1}}}^0 \neq 0$ .  $\square$

**Example 4.** 1) Let  $\hat{G} \subset G$  be a principal  $SL_2$ -subgroup of  $G$  with simple root  $\hat{\alpha}$ . Then  $\hat{\Psi}_w = \emptyset$  for all  $w \in W(1)$  and hence all Weyl group elements of length 1 are fit.

2) Let  $\hat{G} \subset \hat{G}^{\times k} = G$  be the diagonal subgroup in a Cartesian product, we have  $W = \hat{W}^{\times k}$  and for  $w = (w_1, \dots, w_k)$  we have

$$\hat{\Psi}_w = \bigcap_{j=1}^k w_j \hat{\Phi}_{w_j} .$$

One may then compute that

$$\hat{\Psi}_{w_0 w} = \hat{w}_0(\cap_j w_j \hat{\Phi}_{w_j}^c) = w_0(\cap_j \hat{\Phi}_{w_j^{-1}}^c) .$$

Thus  $w$  is fit if and only if we have a disjoint union

$$\hat{\Delta}^+ = \sqcup_j \hat{\Phi}_{w_j^{-1}} .$$

**Lemma 6.26.** *Let  $\lambda \in \Lambda^+$  and  $w \in W^0(\lambda)$ . Let  $\alpha \in \Pi$  be such that  $n_{\lambda,\alpha} \neq 0$  (automatic for regular  $\lambda$ ). Then  $l(ws_\alpha) = l(w) \pm 1$  and the following hold:*

$$\begin{aligned} l(ws_\alpha) &= l(w) + 1 \text{ if and only if } ws_\alpha \in W^-(\lambda); \\ l(ws_\alpha) &= l(w) - 1 \text{ if and only if } ws_\alpha \in W^+(\lambda). \end{aligned}$$

*Proof.* Observe that

$$\iota^*(ws_\alpha\lambda) = \iota^*(w\lambda - n_{\lambda,\alpha}w\alpha) = -n_{\lambda,\alpha}\iota^*(w\alpha).$$

We have  $n_{\lambda,\alpha} > 0$ . The multiplication by a simple reflection either increases or decreases the length by 1, and the sign depends on whether  $w\alpha$  is a positive or a negative root. Now the lemma follows from  $\iota^*(\Delta^\pm) \subset \hat{Q}^\pm$ , which holds with the given choices of Borel and Cartan subgroups.  $\square$

**Theorem 6.27.** *Suppose that  $w \in W$  is a fit element. Then the following hold:*

1) *The set*

$$F_w = \{\lambda \in \Lambda_{\mathbb{R}}^{++} : \iota^*w\lambda = 0\},$$

*if nonempty, is a codimension  $\hat{n}$  face of the  $\hat{G}$  ample cone  $C^{\hat{G}}(X)$ .*

2) *If  $\lambda \in \Lambda^{++} \cap C_w$  is an integral element on that face, then*

$$\text{codim}_X X_{\hat{G}}^{us}(\lambda) = 1.$$

*More precisely, the unstable locus contains the divisors  $\overline{\hat{G}Bx_{w_0ws_\alpha}}$  for  $\alpha \in \Pi \cap \Phi_{w_0w}$ .*

3) *For every integral element  $\lambda \in \Lambda \cap \overline{F}_w$ , we have  $\iota^*(w \cdot m\lambda) = -2\rho$  for all  $m \in \mathbb{N}$ . Furthermore, there exists  $m \in \mathbb{N}$  such that for every  $j \in \mathbb{N}$  there is a surjective pullback in cohomology of line bundles*

$$\varphi_{w \cdot m\lambda}^* : V_\lambda^* = H^{\hat{n}}(X, \mathcal{L}_{w \cdot jm\lambda}) \rightarrow H^{\hat{n}}(\hat{X}, \mathcal{K}_{\hat{X}}) = \mathbb{C},$$

*giving rise to an invariant  $\mathbb{C} = \text{Image}((\varphi_{w \cdot m\lambda}^*)^*) \subset V_{m\lambda}$ .*

**Remark 6.28.** From the results of Ressayre, [R10], and Dimitrov and Roth, [DR09b] it is known that, in the case of a diagonal embedding,  $\hat{G} \subset G = \hat{G}^{\times k}$ , all regular faces of codimension  $\hat{\ell}$  in  $C^{\hat{G}}(X)$  are cohomological, i.e. of the form  $F_w$  for some fit  $w$ .

*Proof.* The statements 1) and 3) follow from the results of [T13a], in view of Lemma 6.25 and our description of  $C^{\hat{G}}(X)$  given in Theorem 6.5.

To prove part 2), let  $\lambda \in \Lambda^{++} \cap F_w$ . With our notation this means  $w \in W^0(\lambda)$ . Let  $\alpha \in \Pi \cap \Phi_{w_0w}$ . Then we have  $w\alpha \in \Delta^+$  and  $l(ws_\alpha) = l(w) + 1 = \hat{n} + 1$ . Furthermore, a direct calculation with inversion sets shows that  $\hat{\Psi}_{w_0ws_\alpha} \subset \hat{\Psi}_{w_0w} = \emptyset$ . Thus

$$\text{codim}_X \hat{G}B_{w_0ws_\alpha} = \hat{n} + 1 - \hat{n} = 1.$$

Let  $j$  be such that  $w\alpha(h_j) \neq 0$  (such  $j$  exists since  $\hat{T}$  contains regular elements of  $T$ ). We have

$$ws_\alpha\lambda(h_j) = w(\lambda - n_{\lambda,\alpha}\alpha)(h_j) = -n_{\lambda,\alpha}w\alpha(h_j) < 0.$$

Hence  $w_0ws_\alpha\lambda(h_j) > 0$ , so that  $w_0ws_\alpha \in W \setminus W^{0-}(\lambda)$ , and we may conclude that  $\hat{G}Bx_{w_0ws_\alpha}$  is unstable. This completes the proof.  $\square$

**Remark 6.29.** The above proof exhibits a more general situation in which the unstable locus contains divisors. Namely, we have

$$ws_\alpha\lambda(h_j) = w\lambda(h_j) - n_{\lambda,\alpha}w\alpha(h_j) = -n_{\lambda,\alpha}w\alpha(h_j) \leq 0.$$

Indeed  $\alpha \in \Pi \cap \Phi_{w_0w}$  implies  $w\alpha \in \Delta^+$  and hence  $w\alpha(h_j) \geq 0$  for all  $j$ . We may deduce that  $\hat{G}B_{w_0ws_\alpha}$  is unstable if and only if  $w\alpha(h_j) < 0$ . Such an  $\alpha$  exists at least under some hypothesis. For instance, for diagonal embeddings with sufficiently many factors  $\hat{G} \subset G = \hat{G}^{\times k}$ ,  $k > \hat{\ell}$ . Indeed, for every fundamental

coweight  $h_j$  there is exactly one simple root  $\hat{\alpha}_j \in \hat{\Pi}$  nonvanishing at  $h_j$ . Hence every component of  $\Pi$  contains exactly one simple root nonvanishing at  $h_j$ . Every inversion set contains a simple root. So a fit Weyl group element  $w = (w_1, \dots, w_k)$  can have at most  $\hat{\ell}$  nontrivial components. As  $k > \hat{\ell}$ ,  $w$  has a trivial component, say  $w_q = 1$ . Let  $\alpha_j$  be the  $j$ -th simple root of the  $q$ -th factor of  $G$  so that  $\alpha_j(h_j) > 0$ . We have  $\alpha_j \in \Pi \cap \Phi_{w_0 w}$  and  $w\alpha_j = \alpha_j$ . Then  $ws_\alpha \lambda(h_j) = -n_{\lambda, \alpha} \alpha(h_j) < 0$  and we obtain an unstable divisor.

**Corollary 6.30.** *Suppose that  $w \in W$  is a fit element such that the set  $F_w$  is nonempty. Then every GIT-class  $C \subset C^{\hat{G}}(X)$ , such that  $\overline{C} \cap F_w \neq \emptyset$ , has unstable locus of codimension 1, i.e.,  $\text{codim}_X X_{\hat{G}}^{us}(C) = 1$ .*

*Proof.* The corollary follows from Theorem 6.27 and Lemma 4.1.5 of [DH98]. This lemma states that, given two GIT-classes  $C_1$  and  $C_2$  containing ample line bundles,  $C_1 \subset \overline{C_2}$  implies  $X_{\hat{G}}^{us}(C_1) \subset X_{\hat{G}}^{us}(C_2)$ . Taking  $C_1 = F_w$ , we get  $\text{codim}_X X_{\hat{G}}^{us}(F_w) = 1$  from Theorem 6.27 and this implies part 1).  $\square$

## 7. MORI CHAMBERS

The goal of this section is to prove theorems 7.3 and 7.4 concerning the structure of the effective cone of any GIT quotient,  $Y$ , of  $X$  defined by a  $\hat{G}$ -movable chamber  $C$  in the  $\hat{G}$ -ample cone  $C^{\hat{G}}(X)$ . By a result of [S14], such a quotient is a Mori dream space, whose pseudoeffective cone is naturally identified with the  $\hat{G}$ -ample cone  $C^{\hat{G}}(X)$ . Here we study the birational geometry  $Y$  and show that the Mori chambers of  $\overline{\text{Eff}}(Y)$  correspond to the GIT-chambers of  $C^{\hat{G}}(X)$ .

We first recall the notion of Mori equivalence for divisors: two big divisors,  $D$  and  $D'$ , on a projective variety  $Y$ , with finitely generated section rings  $R(Y, \mathcal{O}_Y(D))$  and  $R(Y', \mathcal{O}_{Y'}(D'))$  and natural evaluation maps  $f_D : Y \dashrightarrow \text{Proj}(R(Y, \mathcal{O}_Y(D)))$  and  $f_{D'} : Y' \dashrightarrow \text{Proj}(R(Y', \mathcal{O}_{Y'}(D')))$ , are Mori equivalent if there is an isomorphism

$$\varphi : Y_D := \text{Proj}(R(Y, \mathcal{O}_Y(D))) \rightarrow \text{Proj}(R(Y', \mathcal{O}_{Y'}(D'))) =: Y_{D'}$$

making the following diagram commute:

$$\begin{array}{ccc} Y & \xrightarrow{f_D} & Y_D \\ & \searrow f_{D'} & \downarrow \varphi \\ & & Y_{D'} \end{array}$$

(cf. [HK00]). A Mori chamber in the pseudoeffective cone  $\overline{\text{Eff}}(Y)$  is the closure of a full dimensional Mori equivalence class.

We now assume that  $Y = Y_{\lambda_0} = X^{ss}(\lambda_0)/\hat{G}$ , with projection morphism

$$\pi : X^{ss}(\lambda_0) \rightarrow X^{ss}(\lambda_0)/\hat{G},$$

is a quotient such that  $\lambda_0$  belongs to a  $\hat{G}$ -movable chamber in  $C^{\hat{G}}(X)$ .

If  $\lambda \in C^{\hat{G}}(X)$  is a strictly dominant weight for which the line bundle  $\mathcal{L}_\lambda$  on  $X$  descends to a line bundle  $L_\lambda$  on  $Y$ , the section ring  $R(Y, L_\lambda)$  of  $L_\lambda$  is finitely generated, namely  $R(Y, L_\lambda) \cong R(X, \mathcal{L}_\lambda)^{\hat{G}}$ , where the latter ring is finitely generated by the theorem by Hilbert and Nagata. Evaluating homogeneous elements of  $R(Y, L_\lambda)$  in points in  $Y$  outside the stable base locus  $\mathbb{B}(L_\lambda)$  of  $L_\lambda$  then yields a rational map

$$f_\lambda : Y \dashrightarrow Y_\lambda = \text{Proj}(R(Y, L_\lambda)), \quad f_\lambda(y) := \ker \text{ev}_y, \quad y \in Y \setminus \mathbb{B}(L_\lambda),$$

where

$$\mathrm{ev}_y := \bigoplus_{k=1}^{\infty} \mathrm{ev}_{y,k},$$

and

$$\mathrm{ev}_{y,k}(s) := s(y) \in (L_\lambda^k)_y / \mathfrak{m}_y(L_\lambda^k)_y, \quad s \in H^0(Y, L_\lambda^k),$$

where  $\mathfrak{m}_y$  denotes the maximal ideal in the stalk  $\mathcal{O}_{Y,y}$  of the structure sheaf  $\mathcal{O}_Y$  of  $Y$ .

**Lemma 7.1.** *The rational map  $f_\lambda$  is induced by GIT, that is, it is the map*

$$\pi(X^{ss}(\lambda_0) \cap X^{ss}(\lambda)) \rightarrow X^{ss}(\lambda) // \hat{G} = Y_\lambda$$

*induced on quotients by the inclusion  $X^{ss}(\lambda_0) \cap X^{ss}(\lambda) \hookrightarrow X^{ss}(\lambda)$ .*

*Proof.* Since  $\mathcal{L}_\lambda$  is very ample, we can write  $X$  as  $X = \mathrm{Proj}(R(X, \mathcal{L}_\lambda))$ . On the other hand,  $Y_\lambda$  is given by  $Y_\lambda = \mathrm{Proj}(R(X, \mathcal{L}_\lambda)^{\hat{G}})$ . Now, the inclusion  $R(X, \mathcal{L}_\lambda)^{\hat{G}} \hookrightarrow R(X, \mathcal{L}_\lambda)$  yields a rational map of projective spectra

$$q_\lambda : \mathrm{Proj}(R(X, \mathcal{L}_\lambda)) \dashrightarrow \mathrm{Proj}(R(X, \mathcal{L}_\lambda)^{\hat{G}}),$$

given on the level of points by

$$q_\lambda(\mathfrak{p}) := \mathfrak{p} \cap R(X, \mathcal{L}_\lambda)^{\hat{G}}, \quad \mathfrak{p} \in U,$$

where  $U$  is the set of all homogeneous relevant prime ideal for which the homogeneous prime ideals  $\mathfrak{p} \cap R(X, \mathcal{L}_\lambda)^{\hat{G}}$  is relevant, i.e., does not contain  $H^0(X, \mathcal{L}_\lambda^k)^{\hat{G}}$  for all positive integers  $k$ . The closed points of  $U$  are then precisely the points in the semistable locus  $X_{\hat{G}}^{ss}(\lambda)$ . Clearly,  $q_\lambda$  is  $\hat{G}$ -invariant, and we claim that in fact  $q_\lambda = \pi_\lambda$ . Before proving this claim, we show that the claim of the lemma follows from the identity  $q_\lambda = \pi_\lambda$ . Indeed, we can lift  $f_\lambda$  to an evaluation map  $\pi^* f_\lambda : X_{\hat{G}}^{ss}(\lambda_0) \cap X_{\hat{G}}^{ss}(\lambda) \rightarrow \mathrm{Proj}(R(X, \mathcal{L}_\lambda)^{\hat{G}})$  given by

$$x \mapsto (\ker \mathrm{ev}_x) \cap R(X, \mathcal{L}_\lambda), \quad x \in X_{\hat{G}}^{ss}(\lambda_0) \cap X_{\hat{G}}^{ss}(\lambda).$$

If  $F_\lambda : X \rightarrow \mathrm{Proj}(R(X, \mathcal{L}_\lambda))$  denotes the natural map

$$F_\lambda(x) := \ker \mathrm{ev}_x, \quad x \in X,$$

where the evaluation maps  $\mathrm{ev}_x$ , for  $x \in X$ , are defined as above, but for the line bundle  $\mathcal{L}_\lambda$  on  $X$ , the map  $\pi^* f_\lambda$  can be written as the composition

$$(10) \quad \pi^* f_\lambda = q_\lambda \circ F_\lambda|_{X_{\hat{G}}^{ss}(\lambda_0) \cap X_{\hat{G}}^{ss}(\lambda)}.$$

Now, since  $X \cong \mathrm{Proj}(R(X, \mathcal{L}_\lambda))$ , the morphism  $F_\lambda$  providing an isomorphism, we can in fact identify  $F_\lambda$  with the identity morphism of  $X$ . Using this identification, the identity (10) in fact says that  $\pi^* f_\lambda$  is given as the composition of the quotient morphism  $q_\lambda = \pi_\lambda : X_{\hat{G}}^{ss}(\lambda) \rightarrow X_{\hat{G}}^{ss}(\lambda) // \hat{G}$  with the inclusion of the open subsets  $X_{\hat{G}}^{ss}(\lambda_0) \cap X_{\hat{G}}^{ss}(\lambda) \hookrightarrow X_{\hat{G}}^{ss}(\lambda)$ , and this is indeed the claim of the lemma.

We then conclude the proof by showing that  $q_\lambda = \pi_\lambda$ . For this, it suffices to show that  $q_\lambda$  and  $\pi_\lambda$  coincide on open affine subsets defining an open affine  $\hat{G}$ -invariant covering of  $U$ . Let therefore  $s_1, \dots, s_m \in H^0(X, \mathcal{L}_\lambda)^{\hat{G}} = H^0(Y, L_\lambda)$ , for some  $m \in \mathbb{N}$ , be homogeneous generators of the invariant ring  $R(X, \mathcal{L}_\lambda)^{\hat{G}} = R(Y, L_\lambda)$ . (By replacing  $\mathcal{L}_\lambda$  by a power, if necessary, we may without loss of generality assume that this invariant ring has generators in degree one.) Then, putting

$$X_{(s_i)} := \{\mathfrak{p} \in \mathrm{Proj}(R(X, \mathcal{L}_\lambda)) : s_i \notin \mathfrak{p}\} \subseteq X,$$

$$Y_{(s_i)} := \{\mathfrak{p} \in \mathrm{Proj}(R(Y, L_\lambda)) : s_i \notin \mathfrak{p}\} \subseteq Y,$$

for  $i = 1, \dots, m$ , and recalling that these open subsets are affine, namely

$$X_{(s_i)} \cong \mathrm{Spec}(R(X, \mathcal{L}_\lambda)_{(s_i)}), \quad Y_{(s_i)} \cong \mathrm{Spec}(R(Y, L_\lambda)_{(s_i)}),$$

where

$$R(X, \mathcal{L}_\lambda)_{(s_i)} = \left\{ \frac{s}{s_i^k} : k \in \mathbb{N}, \quad s \in H^0(X, \mathcal{L}_\lambda^k) \right\},$$

$$R(Y, L_\lambda)_{(s_i)} = \left\{ \frac{s}{s_i^k} : k \in \mathbb{N}, \quad s \in H^0(Y, L_\lambda^k) \right\}$$

are the homogeneous localizations of the respective rings with respect to the degree-one element  $s_i$ .

The action of  $\hat{G}$  on  $R(X, \mathcal{L}_\lambda)$  by graded ring automorphisms induces an action on the homogeneous localization  $R(X, \mathcal{L}_\lambda)_{(s_i)}$  given by

$$g\left(\frac{s}{s_i^k}\right) := \frac{g(s)}{s_i^k}, \quad g \in \hat{G}, \quad s \in H^0(X, \mathcal{L}_\lambda^k), \quad k \in \mathbb{N},$$

and for this action we clearly have

$$(R(X, \mathcal{L}_\lambda)_{(s_i)})^{\hat{G}} = (R(X, \mathcal{L}_\lambda)^{\hat{G}})_{(s_i)} = R(Y, L_\lambda)_{(s_i)},$$

that is, the operation of taking  $\hat{G}$ -invariants commutes with homogeneous localization with respect to  $s_i$ . In other words, the inclusion

$$(11) \quad R(X, \mathcal{L}_\lambda)_{(s_i)}^{\hat{G}} \hookrightarrow R(X, \mathcal{L}_\lambda)_{(s_i)}$$

of the subring of  $\hat{G}$ -invariants is given by the restriction  $q_\lambda|_{X_{(s_i)}} : X_{(s_i)} \rightarrow Y_{(s_i)}$ . Since the embedding of rings (11) defines a Hilbert quotient, the uniqueness of good quotients implies that it coincides with the restriction of  $\pi_\lambda$  to  $X_{(s_i)}$ . This shows that  $q_\lambda = \pi_\lambda$ .  $\square$

**Lemma 7.2.** *Let  $\lambda, \lambda' \in C^{\hat{G}}(X)$  be Mori equivalent dominant weights each belonging to some GIT-chamber. Then the semistable loci  $X^{ss}(\lambda)$  and  $X^{ss}(\lambda')$  are equal in codimension one, that is, they coincide outside a closed subset of  $X$  of codimension at least two.*

*Proof.* Let  $f_\lambda : Y \dashrightarrow Y_\lambda$  and  $f_{\lambda'} : Y \dashrightarrow Y_{\lambda'}$  be the rational maps defined by the line bundles on  $Y$ . By assumption, there is an isomorphism  $\varphi : Y_\lambda \rightarrow Y_{\lambda'}$  yielding a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{f_\lambda} & Y_\lambda \\ & \searrow f_{\lambda'} & \downarrow \varphi \\ & & Y_{\lambda'}, \end{array}$$

from which it follows that  $\text{exc}(f_\lambda) = \text{exc}(f_{\lambda'})$ . Since  $\lambda_0, \lambda, \lambda'$  are all in GIT-chambers,  $f_\lambda$  and  $f_{\lambda'}$  define isomorphisms

$$\pi(X^{ss}(\lambda_0) \cap X^{ss}(\lambda)) \xrightarrow{f_\lambda} \pi_\lambda(X^{ss}(\lambda_0) \cap X^{ss}(\lambda)) \subseteq Y_\lambda,$$

$$\pi(X^{ss}(\lambda_0) \cap X^{ss}(\lambda')) \xrightarrow{f_{\lambda'}} \pi_\lambda(X^{ss}(\lambda_0) \cap X^{ss}(\lambda')) \subseteq Y_{\lambda'}.$$

Hence,

$$\text{exc}(f_\lambda) \subseteq \pi(X^{ss}(\lambda_0) \cap X^{us}(\lambda)) = \mathbb{B}(L_\lambda),$$

where  $\mathbb{B}(L_\lambda) \subseteq Y$  is the stable base locus of the line bundle  $L_\lambda$  on  $Y$ , and, similarly,

$$\text{exc}(f_{\lambda'}) \subseteq \pi(X^{ss}(\lambda_0) \cap X^{us}(\lambda')) = \mathbb{B}(L_{\lambda'}).$$

We now claim that any extension  $f : Y \dashrightarrow Y_\lambda$  of the rational map  $f_\lambda$  to some open subset  $O \subseteq Y$  containing  $Y \setminus \mathbb{B}(L_\lambda)$  contracts (the intersection with  $O$  of) every divisorial component of the stable base locus  $\mathbb{B}(L_\lambda)$ . Indeed, by [D01](Lemma 7.10) (and its proof), there exists a birational morphism  $q : \tilde{Y} \rightarrow Y$ , defining an



isomorphism outside  $q^{-1}(\mathbb{B}(L_\lambda))$ , and a birational morphism  $\tilde{f} : \tilde{Y} \rightarrow Y_\lambda$  with  $f_\lambda \circ q = \tilde{f}$ , such that  $\tilde{f}$  contracts all Cartier divisors with support in  $q^{-1}(\mathbb{B}(L_\lambda))$ . Since  $Y$  is a geometric quotient,  $Y$  is  $\mathbb{Q}$ -factorial, and hence  $f$  in fact contracts all divisors with support in  $q^{-1}(\mathbb{B}(L_\lambda))$  that are preimages of divisors in  $\mathbb{B}(L_\lambda)$ .

Since  $q$  is a birational morphism, and  $Y$  is  $\mathbb{Q}$ -factorial, the image in  $Y$  of the exceptional locus  $\text{exc}(q)$  has codimension at least two (cf. [D01, 1.40]), so that  $\tilde{f}$  can be identified with a rational map  $Y \dashrightarrow Y_\lambda$ , defined on the open subset  $Y \setminus q(\text{exc}(q))$ , and this rational map thus also contracts the divisorial components of  $\mathbb{B}(L_\lambda)$ . Since any birational extension  $f : Y \dashrightarrow Y_\lambda$  of  $f_\lambda$  has to agree with  $\tilde{f}$  on any open subset where both maps are defined,  $f$  also contracts the divisorial components of  $\mathbb{B}(L_\lambda)$ .

The above argument also applies to  $f'_\lambda$ , and hence we conclude that all divisorial components in  $\mathbb{B}(L_\lambda) \cup \mathbb{B}(L'_\lambda)$  lie in the exceptional locus  $\text{exc}(f_\lambda) = \text{exc}(f_{\lambda'}) \subseteq \mathbb{B}(L_\lambda) \cap \mathbb{B}(L'_\lambda)$ . Hence,  $\mathbb{B}(L_\lambda)$  and  $\mathbb{B}(L_{\lambda'})$  coincide in codimension one. Since  $\pi : X^{ss}(\lambda_0) \rightarrow Y$  defines a geometric quotient, this implies that the preimages of  $\mathbb{B}(L_\lambda)$  and  $\mathbb{B}(L'_\lambda)$  coincide in codimension one in  $X^{ss}(\lambda_0)$ , i.e.,

$$X^{ss}(\lambda) \cap X^{ss}(\lambda_0) = X^{ss}(\lambda') \cap X^{ss}(\lambda_0)$$

in codimension one. Finally, since the unstable locus  $X^{ss}(\lambda_0)$  is of codimension at least two, it follows that the identity  $X^{ss}(\lambda) = X^{ss}(\lambda')$  holds in codimension one.  $\square$

**Theorem 7.3.** *Assume that  $\lambda_0 \in C^{\hat{G}}(X)$  is a dominant weight belonging to a  $\hat{G}$ -movable chamber, and let  $Y := X^{ss}(\lambda_0)/\hat{G}$  be the corresponding quotient. Then the identification  $C^{\hat{G}}(X) \cong \overline{\text{Eff}}(Y)$  of the  $\hat{G}$ -ample cone of  $X$  with the pseudoeffective cone of  $Y$  yields an identification of the GIT-chambers in  $C^{\hat{G}}(X)$  with the Mori chambers of  $\overline{\text{Eff}}(Y)$ .*

*Moreover, every rational contraction  $f : Y \dashrightarrow Y'$ , where  $Y'$  is a normal projective variety, is induced by GIT, that is,  $Y' = Y_\lambda$ , and  $f = f_\lambda$ , for some  $\lambda \in C^{\hat{G}}(X)$ .*

*Proof.* Assuming that the strictly dominant weights  $\lambda$  and  $\lambda'$  are GIT-equivalent, i.e.,  $X_G^{ss}(\lambda) = X_G^{ss}(\lambda')$ , let

$$\varphi : Y_\lambda = X_G^{ss}(\lambda)/\hat{G} \rightarrow X_G^{ss}(\lambda')/\hat{G} = Y_{\lambda'}$$

be the induced isomorphism of the quotients. The Mori equivalence of  $f_\lambda$  and  $f_{\lambda'}$  via  $\varphi$  then follows readily from the GIT-descriptions of the rational maps  $f_\lambda$  and  $f_{\lambda'}$  (Lemma 7.1).

Assume now that the line bundles  $L_\lambda$  and  $L_{\lambda'}$  on  $Y$ , for  $\lambda, \lambda'$  in the interior of  $C^{\hat{G}}(X)$  are Mori equivalent. Then, we have a commuting diagram

$$\begin{array}{ccc} Y & \xrightarrow{f_\lambda} & Y_\lambda \\ & \searrow f_{\lambda'} & \downarrow \varphi \\ & & Y_{\lambda'}, \end{array}$$

where  $\varphi$  is an isomorphism of varieties. In order to show that  $\lambda$  and  $\lambda'$  are GIT-equivalent, it suffices to show the inclusion  $X_G^{ss}(\lambda) \subseteq X^{ss}(\lambda')$  since the same argument will yield the reverse inclusion. Let therefore  $x \in X^{ss}(\lambda)$ , and put  $y := \pi_\lambda(x)$ ,  $y' := \varphi(y)$ . The description of  $Y_{\lambda'}$  as the quotient  $Y_{\lambda'} = X^{ss}(\lambda')/\hat{G}$ , and the fact that the line bundle  $\mathcal{L}_\lambda$  on  $X$  descends to an ample line bundle  $A'$  on  $Y_{\lambda'}$  shows that there exists a  $\hat{G}$ -invariant section  $\tilde{s}' \in H^0(X, \mathcal{L}_{\lambda'})^{\hat{G}}$  and a section  $s' \in H^0(Y_{\lambda'}, A')$  with  $\tilde{s}'|_{X_G^{ss}(\lambda')} = \pi_{\lambda'}^* s'$ , and  $s'(y') \neq 0$ . Moreover,  $A := \varphi^* A'$  is an

ample line bundle on  $Y_\lambda$ , and the commutativity of the above diagram shows that the identity of line bundles

$$f_\lambda^* A = f_{\lambda'}^* A'$$

holds on the open subset  $Y \cap \pi(X_G^{ss}(\lambda) \cap X_G^{ss}(\lambda'))$  of  $V$ . Hence, we have the identity of line bundles

$$(12) \quad \pi_\lambda^* A = \mathcal{L}_{\lambda'}$$

on the open subset  $O := X_G^{ss}(\lambda_0) \cap X_G^{ss}(\lambda) \cap X_G^{ss}(\lambda')$  of  $X_G^{ss}(\lambda)$  (cf. Lemma 7.1). Now, by Lemma 7.2,  $X_G^{ss}(\lambda) = X_G^{ss}(\lambda')$  in codimension one, so the open subset  $O \subseteq X_G^{ss}(\lambda)$  has a complement of codimension at least two in  $X_G^{ss}(\lambda)$ . Hence, the identity of line bundles (12) holds on all of  $X_G^{ss}(\lambda)$ . In particular, the restriction of the section  $\tilde{s}'$  to  $X_G^{ss}(\lambda)$  defines a section of  $\pi_\lambda^* A$  yielding an extension of the section  $\pi_\lambda^* \varphi^* s'$  (since they coincide on  $X_G^{ss}(\lambda_0) \cap X_G^{ss}(\lambda) \cap X_G^{ss}(\lambda')$ ). Hence,  $\tilde{s}'(x) = (\pi_\lambda^* \varphi^* s')(x) = s(y') \neq 0$ , i.e.,  $x \in X_G^{ss}(\lambda')$ . This shows that  $X_G^{ss}(\lambda) \subseteq X_G^{ss}(\lambda')$ , and hence we have proved the first claim about the identification of Mori chambers with GIT-chambers.

Since  $Y$  is a Mori dream space, the second part concerning rational contractions follows immediately from the identification of  $\overline{\text{Eff}}(Y)$  with  $C^{\hat{G}}(X)$  and the characterization ([HK00, Thm. 2.3]) of rational contractions  $f : Y \dashrightarrow Y'$  onto normal projective varieties  $Y'$  as precisely the rational contractions  $f_D : Y \dashrightarrow \text{Proj}(R(Y, \mathcal{O}_Y(D)))$ , for effective divisors  $D$  on  $Y$ .  $\square$

**Theorem 7.4.** *The quotient  $Y = X^{ss}(C)/\hat{G}$  is a Mori dream space with  $\overline{\text{Eff}}(Y) = C^{\hat{G}}(X)$ . This identification of cones, together with the identification of Mori chambers with GIT-chambers, yields an identification of*

(i) *the nef cone,  $\text{Nef}(Y)$ , of  $Y$  with the closure  $\overline{C}$  of the chamber  $C$ ,*

(ii) *the movable cone,  $\text{Mov}(Y)$ , of  $Y$  with the  $\hat{G}$ -movable cone  $\text{Mov}^{\hat{G}}(X)$ .*

*Proof.* The nef cone, being the closure of a Mori chamber, corresponds to the closure of some GIT-chamber. Since every integral divisor in the chamber  $C$  admits a multiple which descends to an ample divisor on  $Y$ , the chamber  $C$  is the unique chamber corresponding to the nef cone. This proves (i).

For part (ii), if  $D$  is integral divisor on  $Y$ , let  $\pi^* D$  denote the extension to  $X$  of the pullback of  $D$  by the quotient morphism  $\pi : X^{ss}(C)/\hat{G} \rightarrow Y$ . The stable base locus,  $\mathbb{B}(D)$ , of  $D$  is then given by

$$(13) \quad \mathbb{B}(D) = \pi(X^{us}(D) \cap X^{ss}(C)).$$

Since the fibres of  $\pi$  all have the dimension  $\dim \hat{G}$ , and since the unstable locus  $X^{us}(C)$  is of codimension at least two, the identity (13) shows in particular that  $\pi^* D$  is  $\hat{G}$ -movable if  $D$  is movable. Hence,  $\text{Mov}(Y) \subseteq \text{Mov}^{\hat{G}}(X)$ .

Conversely, if  $E$  is an integral  $\hat{G}$ -movable divisor on  $X$ , which we can without loss of generality assume to descend to a divisor  $\pi_*^{\hat{G}}(E)$  on  $Y$ , such that  $\pi^* \pi_*^{\hat{G}}(E) = E$ , the identity (13) applied to  $D := \pi_*^{\hat{G}}(E)$  shows that  $\pi_*^{\hat{G}}(E)$  is movable. Hence we also have the inclusion  $\text{Mov}^{\hat{G}}(X) \subseteq \text{Mov}(Y)$ .  $\square$

## REFERENCES

- [BK06] P. Belkale, S. Kumar, *Eigenvalue problem and a new product in cohomology of flag varieties*, *Inventiones Math.* 166 (2006), 185–228.

- [BKR12] P. Belkale, S. Kumar, N. Ressayre, *A generalization of Fulton’s conjecture for arbitrary groups*, Math. Ann. **354** (2012), 401–425.
- [BS00] A. Berenstein, R. Sjamaar, *Coadjoint orbits, moment polytopes, and the Hilbert-Mumford criterion*, J. of AMS **13** (2000), 433–466.
- [D01] Debarre, O., *Higher-dimensional algebraic geometry*, Universitext, Springer, 2001.
- [DW11] H. Derksen, J. Weyman, *The combinatorics of quiver representations*, Ann. Inst. Fourier (Grenoble) **61** (2011), no. 3, 1061–1131.
- [DR09a] I. Dimitrov, M. Roth, *Geometric realization of PRV components and the Littlewood-Richardson cone*, Symmetry in mathematics and physics, Contemp. Math. **490**, 83–95, AMS, Providence, 2009.
- [DR09b] I. Dimitrov, M. Roth, *Cup products of line bundles on homogeneous varieties and generalized PRV components of multiplicity one*, arXiv:0909.2280v1, 2009.
- [DH98] I.V. Dolgachev, Y. Hu, *Variation of geometric invariant theory quotients*, Pub. IHES **78** (1998), 5–56.
- [FMK94] J. Fogarty, D. Mumford, F. Kirwan, *Geometric invariant theory*, 3rd edn, Springer Verlag, New York, 1994.
- [H82] G.J. Heckman, *Projections of orbits and asymptotic behaviour of multiplicities for compact connected Lie groups*, Invent. Math. **67** (1982) 333–356.
- [HK00] Y. Hu, S. Keel, *Mori dream spaces and GIT*, Michigan Math. J. **48** (2000), 331–348.
- [Kir84] F. C. Kirwan, *Cohomology of quotients in symplectic and algebraic geometry*, Mathematical Notes, Vol. 31, Princeton Univ. Press, 1984.
- [KKV89] F. Knop, H. Kraft, T. Vust, *The Picard group of a  $G$ -variety*, Algebraische Transformationsgruppen und Invariantentheorie, DMV Sem. **13**, 77–87, Birkhäuser, Basel, 1989.
- [Kos61] B. Kostant, *Lie algebra cohomology and the generalized Borel-Weil theorem*, Ann. of Math. (2) **74** (1961), 329–387.
- [N84] L. Ness, *A Stratification of the null cone via the moment map*, Amer. J. of Math. **106** (1984), 1281–1329.
- [R98] N. Ressayre, *An example of a thick wall. Appendix to “Variation of geometric invariant theory quotients” by Dolgachev and Hu*, Pub. IHES **87** (1998), 53–56.
- [R10] N. Ressayre, *Geometric invariant theory and the generalized eigenvalue problem*, Invent math **180** (2010), 389–441.
- [R12] N. Ressayre, *A cohomology-free description of eigencones in types A, B, and C*, Int. Math. Res. Not. 2012, no. 21, 4966–5005.
- [RR11] N. Ressayre, E. Richmond, *Branching Schubert calculus and the Belkale-Kumar product on cohomology*, Proc. AMS **139** (2011), 835–848.
- [S14] H. Seppänen, *Global branching laws by global Okounkov bodies*, arXiv:1409.2025, 2014.
- [ST15] H. Seppänen, V.V. Tsanov, *Geometric invariant theory for principal three-dimensional subgroups acting on flag varieties*, to appear in the proceedings of the DFG Priority Programme “1388 Representation Theory” conference in Bad Honnef, 2015. arXiv:1503.07105.
- [T96] M. Thaddeus, *Geometric invariant theory and flips*, J. Amer. Math. Soc. **9** (1996), 691–723.
- [T13a] V.V. Tsanov, *Embeddings of semisimple complex Lie groups and cohomological components of modules*, J. of Algebra **373** (2013), 1–29.

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